8-3 Dot Products and Vector Projections

Find the dot product of u and v. Then determine if u and v are orthogonal.

1. $\mathbf{u} = \langle 3, -5\rangle$, $\mathbf{v} = \langle 6, 2\rangle$

SOLUTION:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 3,-5\rangle\cdot\langle 6,2\rangle\\ &= 3(6)+(-5)(2)\\ &= 8\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} \neq 0$, **u** and **v** are not orthogonal.

2. $\mathbf{u} = \langle -10, -16\rangle$, $\mathbf{v} = \langle -8, 5\rangle$

SOLUTION:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle -10,-16\rangle\cdot\langle -8,5\rangle\\ &= -10(-8)+(-16)(5)\\ &= 0\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} = 0$, **u** and **v** are orthogonal.

3. $\mathbf{u} = \langle 9, -3\rangle$, $\mathbf{v} = \langle 1, 3\rangle$

SOLUTION:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 9,-3\rangle\cdot\langle 1,3\rangle\\ &= 9(1)+(-3)(3)\\ &= 0\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} = 0$, **u** and **v** are orthogonal.

4. $\mathbf{u} = \langle 4, -4\rangle$, $\mathbf{v} = \langle 7, 5\rangle$

SOLUTION:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 4,-4\rangle\cdot\langle 7,5\rangle\\ &= 4(7)+(-4)(5)\\ &= 8\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} \neq 0$, **u** and **v** are not orthogonal.

5. $\mathbf{u} = \langle 1, -4\rangle$, $\mathbf{v} = \langle 2, 8\rangle$

SOLUTION:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 1,-4\rangle\cdot\langle 2,8\rangle\\ &= 1(2)+(-4)(8)\\ &= -30\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} \neq 0$, **u** and **v** are not orthogonal.

6. $\mathbf{u} = 11\mathbf{i} + 7\mathbf{j}$; $\mathbf{v} = -7\mathbf{i} + 11\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as $\mathbf{u} = \langle 11,7\rangle$ and $\mathbf{v} = \langle -7,11\rangle$.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 11,7\rangle\cdot\langle -7,11\rangle\\ &= 11(-7)+7(11)\\ &= 0\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} = 0$, **u** and **v** are orthogonal.

7. $\mathbf{u} = \langle -4, 6\rangle$, $\mathbf{v} = \langle -5, -2\rangle$

SOLUTION:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle -4,6\rangle\cdot\langle -5,-2\rangle\\ &= -4(-5)+6(-2)\\ &= 8\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} \neq 0$, **u** and **v** are not orthogonal.

8. $\mathbf{u} = 8\mathbf{i} + 6\mathbf{j}$; $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as

$\mathbf{u} = \langle 8,6\rangle$ and $\mathbf{v} = \langle -1,2\rangle$.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 8,6\rangle\cdot\langle -1,2\rangle\\ &= 8(-1)+6(2)\\ &= 4\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} \neq 0$, **u** and **v** are not orthogonal.

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9. **SPORTING GOODS** The vector $\mathbf{u} = \langle 406, 297\rangle$ gives the numbers of men's basketballs and women's basketballs, respectively, in stock at a sporting goods store. The vector $\mathbf{v} = \langle 27.5, 15\rangle$ gives the prices in dollars of the two types of basketballs, respectively.

a. Find the dot product $\mathbf{u} \cdot \mathbf{v}$.

b. Interpret the result in the context of the problem.

SOLUTION:

a.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 406, 297\rangle \cdot \langle 27.5, 15\rangle \\ &= 406(27.5) + 297(15) \\ &= 15{,}620\end{aligned}$$

b. The product of the number of men's basketballs and the price of one men's basketball is 406 · 27.5 or \$11,165. This is the revenue that can be made by selling all of the men's basketballs. The product of the number of women's basketballs and the price of one women's basketball is 297 · 15 or \$4455. This is the revenue that can be made by selling all of the women's basketballs. The dot product represents the sum of these two numbers. The total revenue that can be made by selling all of the basketballs is \$15,620.

Use the dot product to find the magnitude of the given vector.

10. $\mathbf{m} = \langle -3, 11\rangle$

SOLUTION:

Since $|\mathbf{m}|^2 = \mathbf{m}\cdot\mathbf{m}$, then $|\mathbf{m}| = \sqrt{\mathbf{m}\cdot\mathbf{m}}$.

$$\begin{aligned}|\mathbf{m}| &= \sqrt{\mathbf{m}\cdot\mathbf{m}} \\ |\langle -3, 11\rangle| &= \sqrt{\langle -3, 11\rangle \cdot \langle -3, 11\rangle} \\ &= \sqrt{(-3)^2 + 11^2} \\ &= \sqrt{130} \text{ or about } 11.4\end{aligned}$$

11. $\mathbf{r} = \langle -9, -4\rangle$

SOLUTION:

Since $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r}$, then $|\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$.

$$\begin{aligned}|\mathbf{r}| &= \sqrt{\mathbf{r}\cdot\mathbf{r}} \\ |\langle -9, 4\rangle| &= \sqrt{\langle -9, 4\rangle \cdot \langle -9, 4\rangle} \\ &= \sqrt{(-9)^2 + 4^2} \\ &= \sqrt{97} \text{ or about } 9.8\end{aligned}$$

12. $\mathbf{n} = \langle 6, 12\rangle$

SOLUTION:

Since $|\mathbf{n}|^2 = \mathbf{n}\cdot\mathbf{n}$, then $|\mathbf{n}| = \sqrt{\mathbf{n}\cdot\mathbf{n}}$.

$$\begin{aligned}|\mathbf{n}| &= \sqrt{\mathbf{n}\cdot\mathbf{n}} \\ |\langle 6, 12\rangle| &= \sqrt{\langle 6, 12\rangle \cdot \langle 6, 12\rangle} \\ &= \sqrt{6^2 + 12^2} \\ &= \sqrt{180} \\ &= 6\sqrt{5} \text{ or about } 13.4\end{aligned}$$

13. $\mathbf{v} = \langle 1, -18\rangle$

SOLUTION:

Since $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$, then $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

$$\begin{aligned}|\mathbf{v}| &= \sqrt{\mathbf{v}\cdot\mathbf{v}} \\ |\langle 1, -18\rangle| &= \sqrt{\langle 1, -18\rangle \cdot \langle 1, -18\rangle} \\ &= \sqrt{1^2 + (-18)^2} \\ &= \sqrt{325} \\ &= 5\sqrt{13} \text{ or about } 18.0\end{aligned}$$

14. $\mathbf{p} = \langle -7, -2\rangle$

SOLUTION:

Since $|\mathbf{p}|^2 = \mathbf{p}\cdot\mathbf{p}$, then $|\mathbf{p}| = \sqrt{\mathbf{p}\cdot\mathbf{p}}$.

$$\begin{aligned}|\mathbf{p}| &= \sqrt{\mathbf{p}\cdot\mathbf{p}} \\ |\langle -7, -2\rangle| &= \sqrt{\langle -7, -2\rangle \cdot \langle -7, -2\rangle} \\ &= \sqrt{(-7)^2 + (-2)^2} \\ &= \sqrt{53} \text{ or about } 7.3\end{aligned}$$

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15. $\mathbf{t} = \langle 23, -16\rangle$

SOLUTION:

Since $|\mathbf{t}|^2 = \mathbf{t}\cdot\mathbf{t}$, then $|\mathbf{t}| = \sqrt{\mathbf{t}\cdot\mathbf{t}}$.

$$|\mathbf{t}| = \sqrt{\mathbf{t}\cdot\mathbf{t}}$$

$$|\langle 23,-16\rangle| = \sqrt{\langle 23,-16\rangle\cdot\langle 23,-16\rangle}$$

$$= \sqrt{23^2 + (-16)^2}$$

$$= \sqrt{785} \text{ or about } 28.0$$

Find the angle θ between u and v to the nearest tenth of a degree.

16. $\mathbf{u} = \langle 0, -5\rangle$, $\mathbf{v} = \langle 1, -4\rangle$

SOLUTION:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle 0, -5\rangle\cdot\langle 1, -4\rangle}{|\langle 0, -5\rangle||\langle 1, -4\rangle|}$$

$$\cos\theta = \frac{0(1) + (-5)(-4)}{\sqrt{0^2 + (-5)^2}\sqrt{1^2 + (-4)^2}}$$

$$\cos\theta = \frac{0 + 20}{\sqrt{0 + 25}\sqrt{1 + 16}}$$

$$\cos\theta = \frac{20}{\sqrt{25}\sqrt{17}}$$

$$\cos\theta = \frac{20}{5\sqrt{17}}$$

$$\cos\theta = \frac{4}{\sqrt{17}}$$

$$\theta = \cos^{-1}\frac{4}{\sqrt{17}}$$

$$\theta \approx 14.0^\circ$$

17. $\mathbf{u} = \langle 7, 10\rangle$, $\mathbf{v} = \langle 4, -4\rangle$

SOLUTION:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle 7, 10\rangle\cdot\langle 4, -4\rangle}{|\langle 7, 10\rangle||\langle 4, -4\rangle|}$$

$$\cos\theta = \frac{7\cdot 4 + 10(-4)}{\sqrt{7^2 + 10^2}\sqrt{4^2 + (-4)^2}}$$

$$\cos\theta = \frac{28 + (-40)}{\sqrt{49 + 100}\sqrt{16 + 16}}$$

$$\cos\theta = \frac{-12}{\sqrt{149}\sqrt{32}}$$

$$\cos\theta = \frac{-12}{\sqrt{149}\cdot 4\sqrt{2}}$$

$$\cos\theta = \frac{-12}{4\sqrt{298}}$$

$$\cos\theta = \frac{-3}{\sqrt{298}}$$

$$\theta = \cos^{-1}\frac{-3}{\sqrt{298}}$$

$$\theta \approx 100.0^\circ$$

18. $\mathbf{u} = \langle -2, 4\rangle$, $\mathbf{v} = \langle 2, -10\rangle$

SOLUTION:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle -2, 4\rangle \cdot \langle 2, -10\rangle}{|\langle -2, 4\rangle||\langle 2, -10\rangle|}$$

$$\cos\theta = \frac{(-2)2 + 4(-10)}{\sqrt{(-2)^2 + 4^2}\sqrt{2^2 + (-10)^2}}$$

$$\cos\theta = \frac{-4 + (-40)}{\sqrt{4 + 16}\sqrt{4 + 100}}$$

$$\cos\theta = \frac{-44}{\sqrt{20}\sqrt{104}}$$

$$\cos\theta = \frac{-44}{4\sqrt{130}}$$

$$\cos\theta = \frac{-11}{\sqrt{130}}$$

$$\theta = \cos^{-1}\frac{-11}{\sqrt{130}}$$

$$\theta \approx 164.7^\circ$$

19. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -4\mathbf{i} - 2\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as $\mathbf{u} = \langle -2, 3\rangle$ and $\mathbf{v} = \langle -4, -2\rangle$.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle -2, 3\rangle \cdot \langle -4, -2\rangle}{|\langle -2, 3\rangle||\langle -4, -2\rangle|}$$

$$\cos\theta = \frac{(-2)(-4) + 3(-2)}{\sqrt{(-2)^2 + 3^2}\sqrt{(-4)^2 + (-2)^2}}$$

$$\cos\theta = \frac{8 + (-6)}{\sqrt{4 + 9}\sqrt{16 + 4}}$$

$$\cos\theta = \frac{2}{\sqrt{13}\sqrt{20}}$$

$$\cos\theta = \frac{2}{\sqrt{13}\cdot 2\sqrt{5}}$$

$$\cos\theta = \frac{2}{2\sqrt{65}}$$

$$\cos\theta = \frac{1}{\sqrt{65}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{65}}$$

$$\theta \approx 82.9^\circ$$

20. $\mathbf{u} = \langle -9, 0\rangle$, $\mathbf{v} = \langle -1, -1\rangle$

SOLUTION:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\langle -9,0\rangle\cdot\langle -1,-1\rangle}{|\langle -9,0\rangle||\langle -1,-1\rangle|}$$
$$\cos\theta = \frac{(-9)(-1)+0(-1)}{\sqrt{(-9)^2+0^2}\sqrt{(-1)^1+(-1)^2}}$$
$$\cos\theta = \frac{9+0}{\sqrt{81+0}\sqrt{1+1}}$$
$$\cos\theta = \frac{9}{\sqrt{81}\sqrt{2}}$$
$$\cos\theta = \frac{9}{9\sqrt{2}}$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = \cos^{-1}\frac{1}{\sqrt{2}}$$
$$\theta = 45.0^\circ$$

21. $\mathbf{u} = -\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = -7\mathbf{i} - 3\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as $\mathbf{u} = \langle -1, -3\rangle$ and $\mathbf{v} = \langle -7, -3\rangle$.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\langle -1,-3\rangle\cdot\langle -7,-3\rangle}{|\langle -1,-3\rangle||\langle -7,-3\rangle|}$$
$$\cos\theta = \frac{(-1)(-7)+(-3)(-3)}{\sqrt{(-1^2+(-3))^2}\sqrt{(-7)^2+(-3)^3}}$$
$$\cos\theta = \frac{7+9}{\sqrt{1+9}\sqrt{49+9}}$$
$$\cos\theta = \frac{7+9}{\sqrt{10}\sqrt{58}}$$
$$\cos\theta = \frac{16}{2\sqrt{145}}$$
$$\cos\theta = \frac{8}{\sqrt{145}}$$
$$\theta = \cos^{-1}\frac{8}{\sqrt{145}}$$
$$\theta \approx 48.4^\circ$$

22. $\mathbf{u} = \langle 6, 0\rangle$, $\mathbf{v} = \langle -10, 8\rangle$

SOLUTION:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\langle 6,0\rangle\cdot\langle -10,8\rangle}{|\langle 6,0\rangle||\langle -10,8\rangle|}$$
$$\cos\theta = \frac{6(-10)+0(8)}{\sqrt{6^2+0^2}\sqrt{(-10)^2+8^2}}$$
$$\cos\theta = \frac{-60+0}{\sqrt{36+0}\sqrt{100+64}}$$
$$\cos\theta = \frac{-60}{\sqrt{36}\sqrt{164}}$$
$$\cos\theta = \frac{-60}{12\sqrt{41}}$$
$$\cos\theta = \frac{-5}{\sqrt{41}}$$
$$\theta = \cos^{-1}\frac{-5}{\sqrt{41}}$$
$$\theta \approx 141.3^\circ$$

23. $\mathbf{u} = -10\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 10\mathbf{i} - 5\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as
$\mathbf{u} = \langle -10, 1 \rangle$ and $\mathbf{v} = \langle 10, -5 \rangle$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle -10, 1\rangle \cdot \langle 10, -5\rangle}{|\langle -10, 1\rangle||\langle 10, -5\rangle|}$$

$$\cos\theta = \frac{(-10)10 + 1(-5)}{\sqrt{(-10)^2 + 1^2}\sqrt{10^2 + (-5)^2}}$$

$$\cos\theta = \frac{-100 + (-5)}{\sqrt{100 + 1}\sqrt{100 + 25}}$$

$$\cos\theta = \frac{-100 + (-5)}{\sqrt{101}\sqrt{125}}$$

$$\cos\theta = \frac{-105}{5\sqrt{505}}$$

$$\cos\theta = \frac{-21}{\sqrt{505}}$$

$$\theta = \cos^{-1}\frac{-21}{\sqrt{505}}$$

$$\theta \approx 159.1^\circ$$

24. **CAMPING** Regina and Luis set off from their campsite to search for firewood. The path that Regina takes can be represented by $\mathbf{u} = \langle 3, -5\rangle$. The path that Luis takes can be represented by $\mathbf{v} = \langle -7, 6\rangle$. Find the angle between the pair of vectors.

SOLUTION:

Use the formula for finding the angle between two vectors.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle 3, -5\rangle \cdot \langle -7, 6\rangle}{|\langle 3, -5\rangle||\langle -7, 6\rangle|}$$

$$\cos\theta = \frac{-21 + (-30)}{\sqrt{34}\sqrt{85}}$$

$$\cos\theta = \frac{-51}{17\sqrt{10}}$$

$$\theta = \cos^{-1}\frac{-51}{17\sqrt{10}} \text{ or about } 161.6^\circ$$

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

25. $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j}$, $\mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as
$\mathbf{u} = \langle 3, 6\rangle$ and $\mathbf{v} = \langle -5, 2\rangle$.

Find the projection of **u** onto **v**.

$$\begin{aligned}
\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\
&= \frac{\langle 3, 6\rangle \cdot \langle -5, 2\rangle}{|\langle -5, 2\rangle|^2}\langle -5, 2\rangle \\
&= \frac{-15 + 12}{\left(\sqrt{29}\right)^2}\langle -5, 2\rangle \\
&= \frac{-3}{29}\langle -5, 2\rangle \\
&= \left\langle \frac{15}{29}, -\frac{6}{29}\right\rangle
\end{aligned}$$

To write **u** as the sum of two orthogonal vectors, start by writing **u** as the sum of two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, or $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$. Since one of the vectors is the projection of **u** onto **v**, let $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w}_2$.

$$\begin{aligned}
\mathbf{u} &= \mathbf{w}_1 + \mathbf{w}_2 \\
\mathbf{u} - \mathbf{w}_1 &= \mathbf{w}_2 \\
\langle 3, 6\rangle - \left\langle \frac{15}{29}, -\frac{6}{29}\right\rangle &= \mathbf{w}_2 \\
\left\langle \frac{87}{29}, \frac{174}{29}\right\rangle - \left\langle \frac{15}{29}, -\frac{6}{29}\right\rangle &= \mathbf{w}_2 \\
\left\langle \frac{72}{29}, \frac{180}{29}\right\rangle &= \mathbf{w}_2
\end{aligned}$$

Thus, $\mathbf{u} = \left\langle \frac{15}{29}, -\frac{6}{29}\right\rangle + \left\langle \frac{72}{29}, \frac{180}{29}\right\rangle$.

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26. $\mathbf{u} = \langle 5, 7\rangle$, $\mathbf{v} = \langle -4, 4\rangle$

SOLUTION:

Find the projection of **u** onto **v**.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{\langle 5,7\rangle\cdot\langle -4,4\rangle}{|\langle -4,4\rangle|^2}\langle -4,4\rangle\\ &= \frac{-20+28}{\left(\sqrt{32}\right)^2}\langle -4,4\rangle\\ &= \frac{8}{32}\langle -4,4\rangle\\ &= \frac{1}{4}\langle -4,4\rangle\\ &= \langle -1,1\rangle\end{aligned}$$

To write **u** as the sum of two orthogonal vectors, start by writing **u** as the sum of two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, or $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$. Since one of the vectors is the projection of **u** onto **v**, let $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w}_2$.

$$\begin{aligned}\mathbf{u} &= \mathbf{w}_1 + \mathbf{w}_2\\ \mathbf{u} - \mathbf{w}_1 &= \mathbf{w}_2\\ \langle 5, 7\rangle - \langle -1, 1\rangle &= \mathbf{w}_2\\ \langle 6, 6\rangle &= \mathbf{w}_2\end{aligned}$$

Thus, $\mathbf{u} = \langle -1,1\rangle + \langle 6,6\rangle$.

27. $\mathbf{u} = \langle 8, 2\rangle$, $\mathbf{v} = \langle -4, 1\rangle$

SOLUTION:

Find the projection of **u** onto **v**.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{\langle 8,2\rangle\cdot\langle -4,1\rangle}{|\langle -4,1\rangle|^2}\langle -4,1\rangle\\ &= \frac{-32+2}{\left(\sqrt{17}\right)^2}\langle -4,1\rangle\\ &= \frac{-30}{17}\langle -4,1\rangle\\ &= \left\langle \frac{120}{17}, -\frac{30}{17}\right\rangle\end{aligned}$$

To write **u** as the sum of two orthogonal vectors, start by writing **u** as the sum of two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, or $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$. Since one of the vectors is the projection of **u** onto **v**, let $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w}_2$.

$$\begin{aligned}\mathbf{u} &= \mathbf{w}_1 + \mathbf{w}_2\\ \mathbf{u} - \mathbf{w}_1 &= \mathbf{w}_2\\ \langle 8, 2\rangle - \left\langle \frac{120}{17}, -\frac{30}{17}\right\rangle &= \mathbf{w}_2\\ \left\langle \frac{136}{17}, \frac{34}{17}\right\rangle - \left\langle \frac{120}{17}, -\frac{30}{17}\right\rangle &= \mathbf{w}_2\\ \left\langle \frac{16}{17}, \frac{64}{17}\right\rangle &= \mathbf{w}_2\end{aligned}$$

Thus, $\mathbf{u} = \left\langle \frac{120}{17}, -\frac{30}{17}\right\rangle + \left\langle \frac{16}{17}, \frac{64}{17}\right\rangle$.

28. $\mathbf{u} = 6\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -3\mathbf{i} + 9\mathbf{j}$

SOLUTION:

Write $\mathbf{u}$ and $\mathbf{v}$ in component form as $\mathbf{u} = \langle 6, 1\rangle$ and $\mathbf{v} = \langle -3, 9\rangle$.

Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{\langle 6,1\rangle\cdot\langle -3,9\rangle}{|\langle -3,9\rangle|^2}\langle -3,9\rangle\\ &= \frac{-18+9}{\left(\sqrt{90}\right)^2}\langle -3,9\rangle\\ &= \frac{-9}{90}\langle -3,9\rangle\\ &= -\frac{1}{10}\langle -3,9\rangle\\ &= \left\langle \frac{3}{10}, -\frac{9}{10}\right\rangle\end{aligned}$$

To write $\mathbf{u}$ as the sum of two orthogonal vectors, start by writing $\mathbf{u}$ as the sum of two vectors $\mathbf{w_1}$ and $\mathbf{w_2}$, or $\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$. Since one of the vectors is the projection of $\mathbf{u}$ onto $\mathbf{v}$, let $\mathbf{w_1} = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w_2}$.

$$\begin{aligned}\mathbf{u} &= \mathbf{w_1} + \mathbf{w_2}\\ \mathbf{u} - \mathbf{w_1} &= \mathbf{w_2}\\ \langle 6, 1\rangle - \left\langle \frac{3}{10}, -\frac{9}{10}\right\rangle &= \mathbf{w_2}\\ \left\langle \frac{60}{10}, \frac{10}{10}\right\rangle - \left\langle \frac{3}{10}, -\frac{9}{10}\right\rangle &= \mathbf{w_2}\\ \left\langle \frac{57}{10}, \frac{19}{10}\right\rangle &= \mathbf{w_2}\end{aligned}$$

Thus, $\mathbf{u} = \left\langle \frac{3}{10}, -\frac{9}{10}\right\rangle + \left\langle \frac{57}{10}, \frac{19}{10}\right\rangle$.

29. $\mathbf{u} = \langle 2, 4\rangle$, $\mathbf{v} = \langle -3, 8\rangle$

SOLUTION:

Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{\langle 2,4\rangle\cdot\langle -3,8\rangle}{|\langle -3,8\rangle|^2}\langle -3,8\rangle\\ &= \frac{-6+32}{\left(\sqrt{73}\right)^2}\langle -3,8\rangle\\ &= \frac{26}{73}\langle -3,8\rangle\\ &= \left\langle -\frac{78}{73}, \frac{208}{73}\right\rangle\end{aligned}$$

To write $\mathbf{u}$ as the sum of two orthogonal vectors, start by writing $\mathbf{u}$ as the sum of two vectors $\mathbf{w_1}$ and $\mathbf{w_2}$, or $\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$. Since one of the vectors is the projection of $\mathbf{u}$ onto $\mathbf{v}$, let $\mathbf{w_1} = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w_2}$.

$$\begin{aligned}\mathbf{u} &= \mathbf{w_1} + \mathbf{w_2}\\ \mathbf{u} - \mathbf{w_1} &= \mathbf{w_2}\\ \langle 2, 4\rangle - \left\langle -\frac{78}{73}, \frac{208}{73}\right\rangle &= \mathbf{w_2}\\ \left\langle \frac{146}{73}, \frac{292}{73}\right\rangle - \left\langle -\frac{78}{73}, \frac{208}{73}\right\rangle &= \mathbf{w_2}\\ \left\langle \frac{224}{73}, \frac{84}{73}\right\rangle &= \mathbf{w_2}\end{aligned}$$

Thus, $\mathbf{u} = \left\langle -\frac{78}{73}, \frac{208}{73}\right\rangle + \left\langle \frac{224}{73}, \frac{84}{73}\right\rangle$.

30. $\mathbf{u} = \langle -5, 9\rangle$, $\mathbf{v} = \langle 6, 4\rangle$

SOLUTION:

Find the projection of **u** onto **v**.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{\langle -5,9\rangle\cdot\langle 6,4\rangle}{|\langle 6,4\rangle|^2}\langle 6,4\rangle\\ &= \frac{-30+36}{\left(\sqrt{52}\right)^2}\langle 6,4\rangle\\ &= \frac{6}{52}\langle 6,4\rangle\\ &= \frac{3}{26}\langle 6,4\rangle\\ &= \left\langle \frac{9}{13},\frac{6}{13}\right\rangle\end{aligned}$$

To write **u** as the sum of two orthogonal vectors, start by writing **u** as the sum of two vectors $\mathbf{w_1}$ and $\mathbf{w_2}$, or $\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$. Since one of the vectors is the projection of **u** onto **v**, let $\mathbf{w_1} = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w_2}$.

$$\begin{aligned}\mathbf{u} &= \mathbf{w_1}+\mathbf{w_2}\\ \mathbf{u}-\mathbf{w_1} &= \mathbf{w_2}\\ \langle -5, 9\rangle - \left\langle \frac{9}{13}, \frac{6}{13}\right\rangle &= \mathbf{w_2}\\ \left\langle -\frac{65}{13}, \frac{117}{13}\right\rangle - \left\langle \frac{9}{13}, \frac{6}{13}\right\rangle &= \mathbf{w_2}\\ \left\langle -\frac{74}{13}, \frac{111}{13}\right\rangle &= \mathbf{w_2}\end{aligned}$$

Thus, $\mathbf{u} = \left\langle \frac{9}{13},\frac{6}{13}\right\rangle + \left\langle -\frac{74}{13},\frac{111}{13}\right\rangle$.

31. $\mathbf{u} = 5\mathbf{i} - 8\mathbf{j}$, $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as $\mathbf{u} = \langle 5,-8\rangle$ and $\mathbf{v} = \langle 6,-4\rangle$.

Find the projection of **u** onto **v**.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{\langle 5,-8\rangle\cdot\langle 6,-4\rangle}{|\langle 6,-4\rangle|^2}\langle 6,-4\rangle\\ &= \frac{30+32}{\left(\sqrt{52}\right)^2}\langle 6,-4\rangle\\ &= \frac{62}{52}\langle 6,-4\rangle\\ &= \frac{31}{26}\langle 6,-4\rangle\\ &= \left\langle \frac{93}{13},-\frac{62}{13}\right\rangle\end{aligned}$$

To write **u** as the sum of two orthogonal vectors, start by writing **u** as the sum of two vectors $\mathbf{w_1}$ and $\mathbf{w_2}$, or $\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$. Since one of the vectors is the projection of **u** onto **v**, let $\mathbf{w_1} = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w_2}$.

$$\begin{aligned}\mathbf{u} &= \mathbf{w_1}+\mathbf{w_2}\\ \mathbf{u}-\mathbf{w_1} &= \mathbf{w_2}\\ \langle 5, -8\rangle - \left\langle \frac{93}{13}, -\frac{62}{13}\right\rangle &= \mathbf{w_2}\\ \left\langle \frac{65}{13}, -\frac{104}{13}\right\rangle - \left\langle \frac{93}{13}, -\frac{62}{13}\right\rangle &= \mathbf{w_2}\\ \left\langle -\frac{28}{13}, -\frac{42}{13}\right\rangle &= \mathbf{w_2}\end{aligned}$$

Thus, $\mathbf{u} = \left\langle \frac{93}{13},-\frac{62}{13}\right\rangle + \left\langle -\frac{28}{13},-\frac{42}{13}\right\rangle$.

8-3 Dot Products and Vector Projections

32. $\mathbf{u} = -2\mathbf{i} - 5\mathbf{j}$, $\mathbf{v} = 9\mathbf{i} + 7\mathbf{j}$

SOLUTION:

Write **u** and **v** in component form as $\mathbf{u} = \langle -2, -5 \rangle$ and $\mathbf{v} = \langle 9, 7 \rangle$.

Find the projection of **u** onto **v**.

$$\begin{aligned} \text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\ &= \frac{\langle -2, -5 \rangle \cdot \langle 9, 7 \rangle}{|\langle 9, 7 \rangle|^2}\langle 9, 7 \rangle \\ &= \frac{-18 - 35}{\left(\sqrt{130}\right)^2}\langle 9, 7 \rangle \\ &= -\frac{53}{130}\langle 9, 7 \rangle \\ &= \left\langle -\frac{477}{130}, -\frac{371}{130} \right\rangle \end{aligned}$$

To write **u** as the sum of two orthogonal vectors, start by writing **u** as the sum of two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, or $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$. Since one of the vectors is the projection of **u** onto **v**, let $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ and solve for $\mathbf{w}_2$.

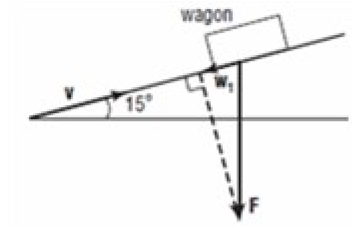
$$\begin{aligned} \mathbf{u} &= \mathbf{w}_1 + \mathbf{w}_2 \\ \mathbf{u} - \mathbf{w}_1 &= \mathbf{w}_2 \\ \langle -2, -5 \rangle - \left\langle -\frac{477}{130}, -\frac{371}{130} \right\rangle &= \mathbf{w}_2 \\ \left\langle -\frac{260}{130}, -\frac{650}{130} \right\rangle - \left\langle -\frac{477}{130}, -\frac{371}{130} \right\rangle &= \mathbf{w}_2 \\ \left\langle \frac{217}{130}, -\frac{279}{130} \right\rangle &= \mathbf{w}_2 \end{aligned}$$

Thus, $\mathbf{u} = \left\langle -\frac{477}{130}, -\frac{371}{130} \right\rangle + \left\langle \frac{217}{130}, -\frac{279}{130} \right\rangle$.

33. **WAGON** Malcolm is pulling his sister in a wagon up a small slope at an incline of 15°. If the combined weight of Malcolm's sister and the wagon is 78 pounds, what force is required to keep her from rolling down the slope?

SOLUTION:

The combined weight of Malcolm's sister and the wagon is the force exerted due to gravity, $\mathbf{F} = \langle 0, -78 \rangle$. To find the force $-\mathbf{w}_1$ required to keep her from sliding down the slope, project **F** onto a unit vector **v** in the direction of the side of the hill.



Find a unit vector **v** in the direction of the side of the hill.

$$\begin{aligned} \mathbf{v} &= \langle |\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta \rangle \\ &= \langle 1(\cos 15^\circ), 1(\sin 15^\circ) \rangle \\ &= \langle 0.966, 0.259 \rangle \end{aligned}$$

Find $\mathbf{w}_1$, the projection of **F** onto the unit vector **v**, $\text{proj}_{\mathbf{v}}\mathbf{F}$.

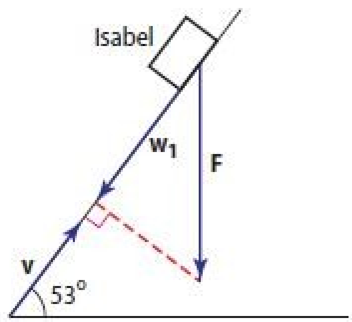
$$\begin{aligned} \mathbf{w}_1 &= \text{proj}_{\mathbf{v}}\mathbf{F} \\ &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\ &= (\mathbf{F} \cdot \mathbf{v})\mathbf{v} \\ &= (\langle 0, -78 \rangle \cdot \langle 0.966, 0.259 \rangle)\mathbf{v} \\ &\approx -20.2\mathbf{v} \end{aligned}$$

Since $\mathbf{w}_1$ points down the hill, the force required is $-\mathbf{w}_1 = -(-20.2\mathbf{v})$ or $20.2\mathbf{v}$. Since **v** is a unit vector, 20.2 pounds represents the magnitude of the force required to keep Malcolm's sister from sliding down the hill.

34. **SLIDE** Isabel is going down a slide but stops herself when she notices that another student is lying hurt at the bottom of the slide. What force is required to keep her from sliding down the slide if the incline is 53° and she weighs 62 pounds?

SOLUTION:

Isabel's weight is the force exerted due to gravity, $\mathbf{F} = \langle 0, -62 \rangle$. To find the force $-\mathbf{w_1}$ required to keep her from sliding down the slope, project **F** onto a unit vector **v** in the direction of the side of the hill.



Find a unit vector **v** in the direction of the side of the hill.

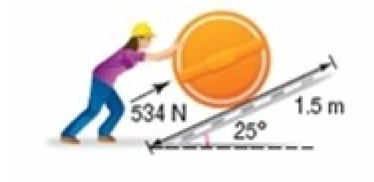
$$\begin{aligned}\mathbf{v} &= \langle |\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta \rangle \\ &= \langle 1(\cos 53^\circ), 1(\sin 53^\circ) \rangle \\ &= \langle 0.602, 0.799 \rangle\end{aligned}$$

Find $\mathbf{w_1}$, the projection of **F** onto the unit vector **v**, $\text{proj}_{\mathbf{v}}\mathbf{F}$.

$$\begin{aligned}\mathbf{w_1} &= \text{proj}_{\mathbf{v}}\mathbf{F} \\ &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\ &= (\mathbf{F}\cdot\mathbf{v})\mathbf{v} \\ &= (\langle 0, -62 \rangle \cdot \langle 0.602, 0.799 \rangle)\mathbf{v} \\ &\approx -49.5\mathbf{v}\end{aligned}$$

Since $\mathbf{w_1}$ points down the hill, the force required is $-\mathbf{w_1} = -(-49.5\mathbf{v})$ or $49.5\mathbf{v}$. Since **v** is a unit vector, 49.5 pounds represents the magnitude of the force required to keep Isabel from sliding down the slide.

35. **PHYSICS** Alexa is pushing a construction barrel up a ramp 1.5 meters long into the back of a truck. She is using a force of 534 newtons and the ramp is 25° from the horizontal. How much work in joules is Alexa doing?



SOLUTION:

Use the projection formula for work. Since the force vector **F** that represents Alexa pushing the construction barrel up the ramp is parallel to $\overrightarrow{AB}$, **F** does not need to be projected on $\overrightarrow{AB}$. So, $\mathbf{F} = \text{proj}_{\overrightarrow{AB}}\mathbf{F}$. The magnitude of the directed distance $\overrightarrow{AB}$ is 1.5.

$$\begin{aligned}W &= \left|\text{proj}_{\overrightarrow{AB}}\mathbf{F}\right|\left|\overrightarrow{AB}\right| \\ &= |\mathbf{F}|\left|\overrightarrow{AB}\right| \\ &= (534)(1.5) \\ &= 801\end{aligned}$$

Verify the result using the dot product formula for work. The component form of the force vector **F** in terms of magnitude and direction angle given is $\langle 534\cos 25^\circ, 534\sin 25^\circ \rangle$. The component form of the directed distance the barrel is moved in terms of magnitude and direction angle given is $\langle 1.5\cos 25^\circ, 1.5\sin 25^\circ \rangle$.

$$\begin{aligned}W &= \mathbf{F}\cdot\overrightarrow{AB} \\ &= \langle 534\cos 25^\circ, 534\sin 25^\circ \rangle \cdot \langle 1.5\cos 25^\circ, 1.5\sin 25^\circ \rangle \\ &= (534\cos 25^\circ)(1.5\cos 25^\circ) + (534\sin 25^\circ)(1.5\sin 25^\circ) \\ &= 801\end{aligned}$$

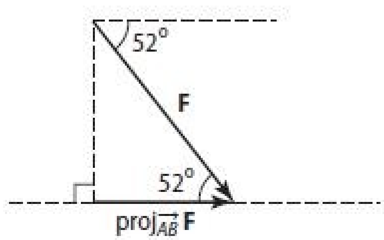
The work that Alexa is doing is 801 joules.

8-3 Dot Products and Vector Projections

36. **SHOPPING** Sophia is pushing a shopping cart with a force of 125 newtons at a downward angle, or angle of depression, of 52°. How much work in joules would Sophia do if she pushed the shopping cart 200 meters?

SOLUTION:

Diagram the situation.



Use the projection formula for work. The magnitude of the projection of **F** onto $\overrightarrow{AB}$ is $|\mathbf{F}|\cos\theta = 152\cos 52^\circ$. The magnitude of the directed distance $\overrightarrow{AB}$ is 200.

$$\begin{aligned} W &= \left|\text{proj}_{\overrightarrow{AB}}\mathbf{F}\right|\left|\overrightarrow{AB}\right| \\ &= (125\cos 52^\circ)(200) \\ &\approx 15{,}391.5 \end{aligned}$$

Therefore, Sophia does about 15,391.5 joules of work pushing the shopping cart.

Find a vector orthogonal to each vector.

37. $\langle -2, -8 \rangle$

SOLUTION:

Sample answer: Two vectors are orthogonal if and only if their dot product is equal to 0. Let $\mathbf{a} = \langle -2, -8 \rangle$ and $\mathbf{b} = \langle x, y \rangle$. Find the dot product of **a** and **b**.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \langle -2, -8 \rangle \cdot \langle x, y \rangle \\ &= -2x + (-8)y \\ &= -2x - 8y \end{aligned}$$

If **a** and **b** are orthogonal, then $-2x - 8y = 0$. Solve for y.

$$\begin{aligned} -2x - 8y &= 0 \\ -2x &= 8y \\ -\frac{x}{4} &= y \end{aligned}$$

Substitute a value for x and solve for y. A value of x that is divisible by 4 will produce an integer value for y. Let $x = -12$.

$$\begin{aligned} -\frac{x}{4} &= y \\ -\frac{(-12)}{4} &= y \\ 3 &= y \end{aligned}$$

A vector orthogonal to $\langle -2, -8 \rangle$ is $\langle -12, 3 \rangle$.

8-3 Dot Products and Vector Projections

38. $\langle 3, 5\rangle$

SOLUTION:

Sample answer: Two vectors are orthogonal if and only if their dot product is equal to 0. Let $\mathbf{a} = \langle 3,5\rangle$ and $\mathbf{b} = \langle x, y\rangle$. Find the dot product of **a** and **b**.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \langle 3,5\rangle\cdot\langle x,y\rangle\\ &= 3x+5y\end{aligned}$$

If **a** and **b** are orthogonal, then $3x + 5y = 0$. Solve for y.

$$\begin{aligned}3x+5y&=0\\ 3x&=-5y\\ -\frac{3x}{5}&=y\end{aligned}$$

Substitute a value for x and solve for y. A value of x that results in a value for $3x$ that is divisible by 5 will produce an integer value for y. Let $x = 10$.

$$\begin{aligned}-\frac{3x}{5}&=y\\ -\frac{3(10)}{5}&=y\\ -6&=y\end{aligned}$$

A vector orthogonal to $\langle 3,5\rangle$ is $\langle 10,-6\rangle$.

39. $\langle 7, -4\rangle$

SOLUTION:

Sample answer: Two vectors are orthogonal if and only if their dot product is equal to 0. Let $\mathbf{a} = \langle 7,-4\rangle$ and $\mathbf{b} = \langle x, y\rangle$. Find the dot product of **a** and **b**.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \langle 7,-4\rangle\cdot\langle x,y\rangle\\ &= 7x+(-4)y\\ &= 7x-4y\end{aligned}$$

If **a** and **b** are orthogonal, then $7x - 4y = 0$. Solve for y.

$$\begin{aligned}7x-4y&=0\\ 7x&=4y\\ \frac{7x}{4}&=y\end{aligned}$$

Substitute a value for x and solve for y. A value of x that results in a value for $7x$ that is divisible by 4 will produce an integer value for y. Let $x = 8$.

$$\begin{aligned}\frac{7x}{4}&=y\\ \frac{7(8)}{4}&=y\\ 14&=y\end{aligned}$$

A vector orthogonal to $\langle 7,-4\rangle$ is $\langle 8,14\rangle$.

40. $\langle -1, 6\rangle$

SOLUTION:

Sample answer: Two vectors are orthogonal if and only if their dot product is equal to 0. Let $\mathbf{a} = \langle -1, 6\rangle$ and $\mathbf{b} = \langle x, y\rangle$. Find the dot product of **a** and **b**.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \langle -1, 6\rangle\cdot\langle x, y\rangle\\ &= (-1)x + 6y\\ &= -x + 6y\end{aligned}$$

If **a** and **b** are orthogonal, then $-x + 6y = 0$. Solve for y.

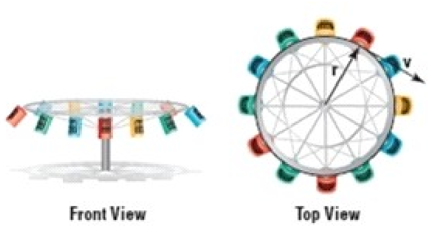
$$\begin{aligned}-x + 6y &= 0\\ -x &= -6y\\ \frac{x}{6} &= y\end{aligned}$$

Substitute a value for x and solve for y. A value of x that is divisible by 6 will produce an integer value for y. Let $x = 6$.

$$\begin{aligned}\frac{x}{6} &= y\\ \frac{(6)}{6} &= y\\ 1 &= y\end{aligned}$$

A vector orthogonal to $\langle -1, 6\rangle$ is $\langle 6, 1\rangle$.

41. **RIDES** For a circular amusement park ride, the position vector **r** is perpendicular to the tangent velocity vector **v** at any point on the circle, as shown below.

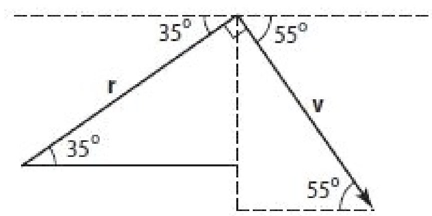


a. If the radius of the ride is 20 feet and the speed of the ride is constant at 40 feet per second, write the component forms of the position vector **r** and the tangent velocity vector **v** when **r** is at a directed angle of 35°.

b. What method can be used to prove that the position vector and the velocity vector that you developed in part **a** are perpendicular? Show that the two vectors are perpendicular.

SOLUTION:

a. Diagram the situation.



Drawing is not to scale.

The component form of **r** can be found using its magnitude and directed angle.

$$\begin{aligned}\mathbf{r} &= \langle |\mathbf{r}|\cos\theta, |\mathbf{r}|\sin\theta\rangle\\ &= \langle 20\cos 35^\circ, 20\sin 35^\circ\rangle\\ &= \langle 16.38, 11.47\rangle\end{aligned}$$

The component form of **v** can be found using its magnitude and the 55° angle. Since the direction of the vector is pointing down, the vertical component will be negative.

$$\begin{aligned}\mathbf{v} &= \langle |\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta\rangle\\ &= \langle 40\cos 55^\circ, -40\sin 55^\circ\rangle\\ &= \langle 22.94, -32.77\rangle\end{aligned}$$

So, $\mathbf{r} = \langle 16.38, 11.47\rangle$ and $\mathbf{v} = \langle 22.94, -32.77\rangle$.

b. If the dot product of the two vectors is equal to 1, then the two vectors are orthogonal or perpendicular. To avoid rounding error, use the exact expression for the components of the vectors found in part **a.**

$$\begin{aligned}\mathbf{r}\cdot\mathbf{v} &= \langle 20\cos 35^\circ, 20\sin 35^\circ\rangle\cdot\langle 40\cos 55^\circ, -40\sin 55^\circ\rangle\\ &= (20\cos 35^\circ)(40\cos 55^\circ) + (20\sin 35^\circ)(-40\sin 55^\circ)\\ &= 0\end{aligned}$$

Given v and u · v, find u.

42. **v** = $\langle 3, -6\rangle$, **u** · **v** = 33

SOLUTION:

Sample answer: Let $\mathbf{u} = \langle x, y\rangle$. Substitute **u**, **v**, and $\mathbf{u} \cdot \mathbf{v}$ into the equation for a dot product.

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \langle x, y\rangle \cdot \langle 3, -6\rangle \\ 33 &= x(3) + y(-6) \\ 33 &= 3x - 6y\end{aligned}$$

Solve for y.

$$\begin{aligned}33 &= 3x - 6y \\ 33 - 3x &= -6y \\ \frac{11 - x}{-2} &= y\end{aligned}$$

Substitute a value for x and solve for y. A value of x that results in a value for $11 - x$ that is divisible by -2 will produce an integer value for y. Let $x = 5$.

$$\begin{aligned}\frac{11 - x}{-2} &= y \\ \frac{11 - 5}{-2} &= y \\ -3 &= y\end{aligned}$$

Therefore, $\mathbf{u} = \langle 5, -3\rangle$.

43. **v** = $\langle 4, 6\rangle$, **u** · **v** = 38

SOLUTION:

Sample answer: Let $\mathbf{u} = \langle x, y\rangle$. Substitute **u**, **v**, and $\mathbf{u} \cdot \mathbf{v}$ into the equation for a dot product.

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \langle x, y\rangle \cdot \langle 4, 6\rangle \\ 38 &= x(4) + y(6) \\ 38 &= 4x + 6y\end{aligned}$$

Solve for y.

$$\begin{aligned}38 &= 4x + 6y \\ 38 - 4x &= 6y \\ \frac{19 - 2x}{3} &= y\end{aligned}$$

Substitute a value for x and solve for y. A value of x that results in a value for $19 - 2x$ that is divisible by 3 will produce an integer value for y. Let $x = -1$.

$$\begin{aligned}\frac{19 - 2x}{3} &= y \\ \frac{19 - 2(-1)}{3} &= y \\ \frac{21}{3} &= y \\ 7 &= y\end{aligned}$$

Therefore, $\mathbf{u} = \langle -1, 7\rangle$.

44. $\mathbf{v} = \langle -5, -1\rangle$, $\mathbf{u} \cdot \mathbf{v} = -8$

SOLUTION:

Sample answer: Let $\mathbf{u} = \langle x, y\rangle$. Substitute **u**, **v**, and $\mathbf{u}\cdot\mathbf{v}$ into the equation for a dot product.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle x, y\rangle\cdot\langle -5, -1\rangle\\ -8 &= x(-5) + y(-1)\\ -8 &= -5x - y\end{aligned}$$

Solve for y.

$$\begin{aligned}-8 &= -5x - y\\ -8 + 5x &= -y\\ 8 - 5x &= y\end{aligned}$$

Substitute a value for x and solve for y. Let $x = 1$.

$$\begin{aligned}8 - 5x &= y\\ 8 - 5(1) &= y\\ 3 &= y\end{aligned}$$

Therefore, $\mathbf{u} = \langle 1, 3\rangle$.

45. $\mathbf{v} = \langle -2, 7\rangle$, $\mathbf{u} \cdot \mathbf{v} = -43$

SOLUTION:

Sample answer: Let $\mathbf{u} = \langle x, y\rangle$. Substitute **u**, **v**, and $\mathbf{u}\cdot\mathbf{v}$ into the equation for a dot product.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle x, y\rangle\cdot\langle -2, 7\rangle\\ -43 &= x(-2) + y(7)\\ -43 &= -2x + 7y\end{aligned}$$

Solve for y.

$$\begin{aligned}-43 &= -2x + 7y\\ -43 + 2x &= 7y\\ \frac{-43 + 2x}{7} &= y\end{aligned}$$

Substitute a value for x and solve for y. A value of x that results in a value for $-43 + 2x$ that is divisible by 7 will produce an integer value for y. Let $x = 4$.

$$\begin{aligned}\frac{-43 + 2x}{7} &= y\\ \frac{-43 + 2(4)}{7} &= y\\ \frac{-35}{7} &= y\\ -5 &= y\end{aligned}$$

Therefore, $\mathbf{u} = \langle 4, -5\rangle$.

8-3 Dot Products and Vector Projections

46. **SCHOOL** A student rolls her backpack from her Chemistry classroom to her English classroom using a force of 175 newtons.

a. If she exerts 3060 joules to pull her backpack 31 meters, what is the angle of her force?
b. If she exerts 1315 joules at an angle of 60°, how far did she pull her backpack?

SOLUTION:

a. Use the projection formula for work. The magnitude of the projection of **F** onto $\overrightarrow{AB}$ is $|\mathbf{F}|\cos\theta = 175\cos\theta$. The magnitude of the directed distance $\overrightarrow{AB}$ is 31 and the work exerted is 3060 joules. Solve for θ.

$$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}||\overrightarrow{AB}|$$
$$3060 = (175\cos\theta)(31)$$
$$0.564 \approx \cos\theta$$
$$\cos^{-1} 0.564 \approx \theta$$
$$55.7^\circ \approx \theta$$

Therefore, the angle of the student's force is about 55.7°.

b. Use the projection formula for work. The magnitude of the projection of **F** onto $\overrightarrow{AB}$ is $|\mathbf{F}|\cos\theta = 175\cos 60^\circ$. The work exerted is 1315 joules. Solve for $\overrightarrow{AB}$.

$$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}||\overrightarrow{AB}|$$
$$1315 = (175\cos 60^\circ)|\overrightarrow{AB}|$$
$$\frac{1315}{(175\cos 60^\circ)} = |\overrightarrow{AB}|$$
$$15.0 \approx |\overrightarrow{AB}|$$

Therefore, the student pulled her backpack about 15 meters.

Determine whether each pair of vectors are *parallel*, *perpendicular*, or *neither*. Explain your reasoning.

47. $\mathbf{u} = \left\langle -\frac{2}{3}, \frac{3}{4} \right\rangle$, $\mathbf{v} = \langle 9, 8 \rangle$

SOLUTION:

Find the angle between **u** and **v**.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\left\langle -\frac{2}{3}, \frac{3}{4} \right\rangle \cdot \langle 9, 8 \rangle}{\left|\left\langle -\frac{2}{3}, \frac{3}{4} \right\rangle\right| |\langle 9, 8 \rangle|}$$
$$\cos\theta = \frac{-6+6}{\sqrt{\frac{145}{144}}\sqrt{145}}$$
$$\cos\theta = \frac{0}{\sqrt{\frac{145}{144}}\sqrt{145}}$$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, the two vectors are orthogonal (perpendicular). The equation can be further worked out to verify this conclusion.

$$\cos\theta = \frac{0}{\sqrt{\frac{145}{144}}\sqrt{145}}$$
$$\cos\theta = 0$$
$$\theta = 90^\circ$$

Perpendicular; sample answer: Since $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are perpendicular.

48. **u** = $\langle -1, -4\rangle$, **v** = $\langle 3, 6\rangle$

SOLUTION:

Find the angle between **u** and **v**.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\langle -1,-4\rangle\cdot\langle 3,6\rangle}{|\langle -1,-4\rangle||\langle 3,6\rangle|}$$
$$\cos\theta = \frac{-3+(-24)}{\sqrt{17}\sqrt{45}}$$
$$\cos\theta = \frac{-27}{3\sqrt{85}}$$
$$\theta = \cos^{-1}\frac{-27}{3\sqrt{85}} \text{ or about } 167.5^\circ$$

The pair of vectors are neither parallel nor perpendicular. Using the formula for the angle between two vectors, $\theta \approx 167.5°$.

49. **u** = $\langle 5, 7\rangle$, **v** = $\langle -15, -21\rangle$

SOLUTION:

Find the angle between **u** and **v**.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\langle 5,7\rangle\cdot\langle -15,-21\rangle}{|\langle 5,7\rangle||\langle -15,-21\rangle|}$$
$$\cos\theta = \frac{-75+(-147)}{\sqrt{74}\sqrt{666}}$$
$$\cos\theta = \frac{-222}{222}$$
$$\cos\theta = -1$$
$$\theta = \cos^{-1}(-1) \text{ or } 180^\circ$$

The vectors are parallel. The angle between the two vectors is 180°. They are headed in opposite directions.

50. **u** = $\langle \sec\theta, \csc\theta\rangle$, **v** = $\langle \csc\theta, -\sec\theta\rangle$

SOLUTION:

Find the angle between **u** and **v**.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\langle \sec\theta,\csc\theta\rangle\cdot\langle \csc\theta,-\sec\theta\rangle}{|\langle \sec\theta,\csc\theta\rangle||\langle \csc\theta,-\sec\theta\rangle|}$$
$$\cos\theta = \frac{\sec\theta\csc\theta+[\csc\theta(-\sec\theta)]}{|\langle \sec\theta,\csc\theta\rangle||\langle \csc\theta,-\sec\theta\rangle|}$$
$$\cos\theta = \frac{\sec\theta\csc\theta-\sec\theta\csc\theta}{|\langle \sec\theta,\csc\theta\rangle||\langle \csc\theta,-\sec\theta\rangle|}$$
$$\cos\theta = \frac{0}{|\langle \sec\theta,\csc\theta\rangle||\langle \csc\theta,-\sec\theta\rangle|}$$

Since **u** · **v** = 0, the two vectors are orthogonal (perpendicular). The equation can be further worked out to verify this conclusion.

$$\cos\theta = \frac{0}{|\langle \sec\theta, \csc\theta\rangle||\langle \csc\theta, -\sec\theta\rangle|}$$
$$\cos\theta = 0$$
$$\theta = 90^\circ$$

Perpendicular; sample answer: Since **u** · **v** = 0, the vectors are perpendicular.

8-3 Dot Products and Vector Projections

Find the angle between the two vectors in radians.

51. $\mathbf{u}=\cos\left(\frac{\pi}{3}\right)\mathbf{i}+\sin\left(\frac{\pi}{3}\right)\mathbf{j}, \mathbf{v}=\cos\left(\frac{3\pi}{4}\right)\mathbf{i}+\sin\left(\frac{3\pi}{4}\right)\mathbf{j}$

SOLUTION:

Simplify each vector.

$$\mathbf{u}=\cos\left(\frac{\pi}{3}\right)\mathbf{i}+\sin\left(\frac{\pi}{3}\right)\mathbf{j} = \frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$$

$$\mathbf{v}=\cos\left(\frac{3\pi}{4}\right)\mathbf{i}+\sin\left(\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}$$

Find the angle between **u** and **v**.

$$\cos\theta=\frac{u\cdot v}{|u||v|}$$

$$\cos\theta=\frac{\left\langle\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle\cdot\left\langle-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\rangle}{\left|\left\langle\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle\right|\left|\left\langle-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\rangle\right|}$$

$$\cos\theta=\frac{-\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}}{\sqrt{\frac{1}{4}+\frac{3}{4}}\sqrt{\frac{2}{4}+\frac{2}{4}}}$$

$$\cos\theta=\frac{-\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}}{1\cdot 1}$$

$$\cos\theta=-\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$$

$$\theta=\cos^{-1}\left(-\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\right)$$

$$\theta=75$$

Thus, the angle between the two vectors is $\frac{75\cdot\pi}{180}=\frac{5\pi}{12}$.

52. $\mathbf{u}=\cos\left(\frac{7\pi}{6}\right)\mathbf{i}+\sin\left(\frac{7\pi}{6}\right)\mathbf{j}, \mathbf{v}=\cos\left(\frac{5\pi}{4}\right)\mathbf{i}+\sin\left(\frac{5\pi}{4}\right)$

SOLUTION:

Simplify each vector.

$$\mathbf{u}=\cos\left(\frac{7\pi}{6}\right)\mathbf{i}+\sin\left(\frac{7\pi}{6}\right)\mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}$$

$$\mathbf{v}=\cos\left(\frac{5\pi}{4}\right)\mathbf{i}+\sin\left(\frac{5\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{j}$$

Find the angle between **u** and **v**.

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta=\frac{\left\langle-\frac{\sqrt{3}}{2},-\frac{1}{2}\right\rangle\cdot\left\langle-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right\rangle}{\left|\left\langle-\frac{\sqrt{3}}{2},-\frac{1}{2}\right\rangle\right|\left|\left\langle-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right\rangle\right|}$$

$$\cos\theta=\frac{\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}}{\sqrt{\frac{3}{4}+\frac{1}{4}}\sqrt{\frac{2}{4}+\frac{2}{4}}}$$

$$\cos\theta=\frac{\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}}{1\cdot 1}$$

$$\cos\theta=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$$

$$\theta=\cos^{-1}\left(\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\right)$$

$$\theta=15$$

Thus, the angle between the two vectors is $\frac{15\cdot\pi}{180}=\frac{\pi}{12}$.

53. $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j},\ \mathbf{v} = \cos\left(\frac{2\pi}{3}\right)\mathbf{i} + \sin\left(\frac{2\pi}{3}\right)\mathbf{j}$

SOLUTION:

Simplify each vector.

$$\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{v} = \cos\left(\frac{2\pi}{3}\right)\mathbf{i} + \sin\left(\frac{2\pi}{3}\right)\mathbf{j} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Find the angle between **u** and **v**.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \cdot \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle}{\left|\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle\right|}$$

$$\cos\theta = \frac{-\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}}{\left|\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle\right|}$$

$$\cos\theta = \frac{0}{\left|\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle\right|}$$

Since **u** · **v** = 0, the two vectors are orthogonal (perpendicular). The equation can be further worked out to verify this conclusion.

$$\cos\theta = \frac{0}{\left|\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle\right|}$$

$$\cos\theta = 0$$

$$\theta = 90^\circ$$

Thus, the angle between the two vectors is $\frac{90\cdot\pi}{180}$ $\frac{\pi}{2}$.

54. $\mathbf{u} = \cos\left(\frac{\pi}{4}\right)\mathbf{i} + \sin\left(\frac{\pi}{4}\right)\mathbf{j},\ \mathbf{v} = \cos\left(\frac{5\pi}{6}\right)\mathbf{i} + \sin\left(\frac{5\pi}{6}\right)\mathbf{j}$

SOLUTION:

Simplify each vector.

$$\mathbf{u} = \cos\left(\frac{\pi}{4}\right)\mathbf{i} + \sin\left(\frac{\pi}{4}\right)\mathbf{j} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{v} = \cos\left(\frac{5\pi}{6}\right)\mathbf{i} + \sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Find the angle between **u** and **v**.

$$\cos\theta = \frac{u\cdot v}{|u||v|}$$

$$\cos\theta = \frac{\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle \cdot \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle}{\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle\right|\left|\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right|}$$

$$\cos\theta = \frac{-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}}{\sqrt{\frac{2}{4}+\frac{2}{4}}\sqrt{\frac{3}{4}+\frac{1}{4}}}$$

$$\cos\theta = \frac{-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}}{1\cdot 1}$$

$$\cos\theta = -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\theta = \cos^{-1}\left(-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)$$

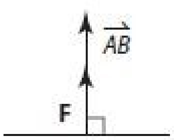
$$\theta = 105$$

Thus, the angle between the two vectors is $\frac{105\cdot\pi}{180} = \frac{7\pi}{12}$.

55. **WORK** Tommy lifts his nephew, who weighs 16 kilograms, a distance of 0.9 meter. The force of weight in newtons can be calculated using $F = mg$, where m is the mass in kilograms and g is 9.8 meters per second squared. How much work did Tommy do to lift his nephew?

SOLUTION:

The force that Tommy is using to lift his nephew is $F = 16 \cdot 9.8$ or 156.8 newtons.
Diagram the situation.



Use the projection formula for work. Since the force vector $\mathbf{F}$ that represents Tommy lifting his nephew is parallel to $\overrightarrow{AB}$, $\mathbf{F}$ does not need to be projected on $\overrightarrow{AB}$. So, $\mathbf{F} = \text{proj}_{\overrightarrow{AB}}\mathbf{F}$. The magnitude of the directed distance $\overrightarrow{AB}$ is 0.9.

$$\begin{aligned} W &= \left|\text{proj}_{\overrightarrow{AB}}\mathbf{F}\right|\left|\overrightarrow{AB}\right| \\ &= |\mathbf{F}|\left|\overrightarrow{AB}\right| \\ &= (156.8)(0.9) \\ &= 141.12 \end{aligned}$$

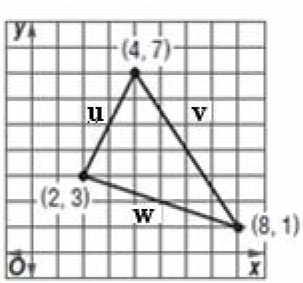
The work that Tommy did to lift his nephew was about 141.1 joules.

The vertices of a triangle on the coordinate plane are given. Find the measures of the angles of each triangle using vectors. Round to the nearest tenth of a degree.

56. (2, 3), (4, 7), (8, 1)

SOLUTION:

Graph the triangle. Label **u**, **v**, and **w**.



The vectors can go in either direction, so find the vectors in component form for each initial point.

$$\mathbf{u} = \langle 2, 4\rangle \text{ or } \mathbf{u} = \langle -2, -4\rangle$$
$$\mathbf{v} = \langle 4, -6\rangle \text{ or } \mathbf{v} = \langle -4, 6\rangle$$
$$\mathbf{w} = \langle -6, 2\rangle \text{ or } \mathbf{w} = \langle 6, -2\rangle$$

Find the angle between **u** and **v**. The vertex (4, 7) must be the initial point of both vectors. Thus, use $\mathbf{u} = \langle -2, -4\rangle$ and $\mathbf{v} = \langle 4, -6\rangle$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos\theta = \frac{\langle -2, -4\rangle \cdot \langle 4, -6\rangle}{|\langle -2, -4\rangle||\langle 4, -6\rangle|}$$
$$\cos\theta = \frac{-8 + 24}{\sqrt{20}\sqrt{52}}$$
$$\cos\theta = \frac{16}{2\sqrt{5} \cdot 2\sqrt{13}}$$
$$\cos\theta = \frac{4}{\sqrt{65}}$$
$$\theta = \cos^{-1}\frac{4}{\sqrt{65}}$$
$$\theta \approx 60.3$$

Find the angle between **v** and **w**. The vertex (8, 1) must be the initial point of both vectors. Thus, use $\mathbf{v} = \langle -4, 6\rangle$ and $\mathbf{w} = \langle -6, 2\rangle$.

$$\cos\theta = \frac{v \cdot w}{|v||w|}$$
$$\cos\theta = \frac{\langle -4, 6\rangle \cdot \langle -6, 2\rangle}{|\langle -4, 6\rangle||\langle -6, 2\rangle|}$$
$$\cos\theta = \frac{24 + 12}{\sqrt{52}\sqrt{40}}$$
$$\cos\theta = \frac{36}{2\sqrt{13} \cdot 2\sqrt{10}}$$
$$\cos\theta = \frac{9}{\sqrt{130}}$$
$$\theta = \cos^{-1}\frac{9}{\sqrt{130}}$$
$$\theta \approx 37.9°$$

180 – (60.3 + 37.9) = 81.8

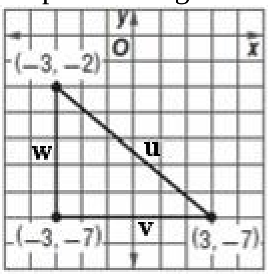
8-3 Dot Products and Vector Projections

The three angles are about 37.9°, 60.3°, and 81.8°.

57. (–3, –2), (–3, –7), (3, –7)

SOLUTION:

Graph the triangle. Label **u**, **v**, and **w**.



The vectors can go either directions, so find the vectors in component form for each initial point.

$\mathbf{u} = \langle 6, -5 \rangle$ or $\mathbf{u} = \langle -6, 5 \rangle$
$\mathbf{v} = \langle -6, 0 \rangle$ or $\mathbf{v} = \langle 6, 0 \rangle$
$\mathbf{w} = \langle 0, 5 \rangle$ or $\mathbf{w} = \langle 0, -5 \rangle$

Find the angle between **u** and **v**. The vertex (3, −7) must be the initial point of both vectors. Thus, use $\mathbf{u} = \langle -6, 5 \rangle$ and $\mathbf{v} = \langle -6, 0 \rangle$.

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

$$\cos\theta = \frac{\langle -6, 5 \rangle \cdot \langle -6, 0 \rangle}{|\langle -6, 5 \rangle||\langle -6, 0 \rangle|}$$

$$\cos\theta = \frac{(-6)(-6) + 5(0)}{\sqrt{(-6)^2 + 5^2} \cdot \sqrt{(-6)^2 + 0^2}}$$

$$\cos\theta = \frac{36 + 0}{\sqrt{61} \cdot 6}$$

$$\cos\theta = \frac{6}{\sqrt{61}}$$

$$\theta = \cos^{-1}\frac{6}{\sqrt{61}}$$

$$\theta \approx 39.8^\circ$$

Find the angle between **v** and **w**. The vertex (−3, −7) must be the initial point of both vectors. Thus, use $\mathbf{v} = \langle 6, 0 \rangle$ and $\mathbf{w} = \langle 0, 5 \rangle$.

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$

$$\cos\theta = \frac{\langle 6, 0 \rangle \cdot \langle 0, 5 \rangle}{|\langle 6, 0 \rangle||\langle 0, 5 \rangle|}$$

$$\cos\theta = \frac{6 \cdot 0 + 0 \cdot 5}{\sqrt{6^2} \cdot \sqrt{5^2}}$$

$$\cos\theta = \frac{0}{6 \cdot 5}$$

$$\cos\theta = 0$$

Since **v** · **w** = 0, the two vectors are perpendicular. The equation can be further worked out to verify this conclusion.

$$\cos\theta = \frac{0}{6 \cdot 5}$$

$$\cos\theta = 0$$

$$\theta = 90^\circ$$

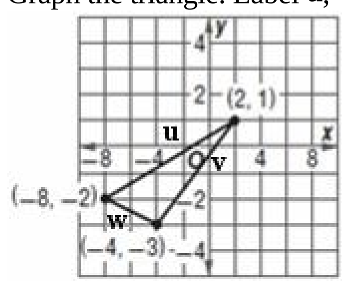
Find the angle between **u** and **w**. The vertex (−3, −2) must be the initial point of both vectors. Thus, use $\mathbf{u} = \langle 6, -5 \rangle$ and $\mathbf{w} = \langle 0, -5 \rangle$.

180 – (90 + 50.2) = 39.8
The three angles are about 39.8°, 50.2°, and 90°.

58. (–4, –3), (–8, –2), (2, 1)

SOLUTION:

Graph the triangle. Label **u**, **v**, and **w**.



The vectors can go in either direction, so find the vectors in component form for each initial point.

$\mathbf{u} = \langle 10, 3 \rangle$ or $\mathbf{u} = \langle -10, -3 \rangle$
$\mathbf{v} = \langle 6, 4 \rangle$ or $\mathbf{v} = \langle -6, -4 \rangle$
$\mathbf{w} = \langle -4, 1 \rangle$ or $\mathbf{w} = \langle 4, -1 \rangle$

Find the angle between **u** and **v**. The vertex (2, 1) must be the initial point of both vectors. Thus, use

$\mathbf{u} = \langle -10, -3 \rangle$ and $\mathbf{v} = \langle -6, -4 \rangle$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle -10, -3 \rangle \cdot \langle -6, -4 \rangle}{|\langle 10, 3 \rangle||\langle -6, -4 \rangle|}$$

$$\cos\theta = \frac{(-10)(-6) + (-3)(-4)}{\sqrt{10^2 + 3^2}\sqrt{(-6)^2 + (-4)^2}}$$

$$\cos\theta = \frac{60 + 12}{\sqrt{109}\sqrt{52}}$$

$$\cos\theta = \frac{36}{\sqrt{1417}}$$

$$\theta = \cos^{-1}\frac{36}{\sqrt{1417}}$$

$$\theta \approx 17.0^\circ$$

Find the angle between **v** and **w**. The vertex (−4, −3) must be the initial point of both vectors. Thus, use $\mathbf{v} = \langle 6, 4 \rangle$ and $\mathbf{w} = \langle -4, 1 \rangle$.

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$

$$\cos\theta = \frac{\langle 6, 4 \rangle \cdot \langle -4, 1 \rangle}{|\langle 6, 4 \rangle||\langle -4, 1 \rangle|}$$

$$\cos\theta = \frac{6(-4) + 4(1)}{\sqrt{6^2 + 4^2}\sqrt{4^2 + 1^2}}$$

$$\cos\theta = \frac{-24 + 4}{\sqrt{52}\sqrt{17}}$$

$$\cos\theta = \frac{-10}{\sqrt{221}}$$

$$\theta = \cos^{-1}\frac{-10}{\sqrt{221}}$$

$$\theta \approx 132.3^\circ$$

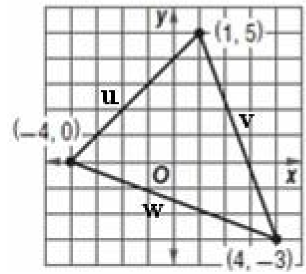
180 – (17.0 + 132.3) = 30.7

Thus, the three angles are about 17.0°, 30.7°, and 132.3°.

59. (1, 5), (4, –3), (–4, 0)

SOLUTION:

Graph the triangle. Label **u**, **v**, and **w**.



The vectors can go in either direction, so find the vectors in component form for each initial point.

$\mathbf{u} = \langle 5, 5 \rangle$ or $\mathbf{u} = \langle -5, -5 \rangle$

$\mathbf{v} = \langle 3, -8 \rangle$ or $\mathbf{v} = \langle -3, 8 \rangle$

$\mathbf{w} = \langle 8, -3 \rangle$ or $\mathbf{w} = \langle -8, 3 \rangle$

Find the angle between **u** and **v**. The vertex (−4, 0) must be the initial point of both vectors. Thus, use $\mathbf{u} = \langle -5, -5 \rangle$ and $\mathbf{v} = \langle 3, -8 \rangle$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{\langle -5, -5 \rangle \cdot \langle 3, -8 \rangle}{|\langle -5, -5 \rangle||\langle 3, -8 \rangle|}$$

$$\cos\theta = \frac{(-5)3 + (-5)(-8)}{\sqrt{(-5)^2 + (-5)^2}\sqrt{3^2 + (-8)^2}}$$

$$\cos\theta = \frac{-15 + 40}{\sqrt{50}\sqrt{73}}$$

$$\cos\theta = \frac{5}{\sqrt{146}}$$

$$\theta = \cos^{-1}\frac{5}{\sqrt{146}}$$

$$\theta \approx 65.6^\circ$$

Find the angle between **v** and **w**. The vertex (4, −3) must be the initial point of both vectors. Thus, use $\mathbf{v} = \langle -3, 8 \rangle$ and $\mathbf{w} = \langle -8, 3 \rangle$.

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$
$$\cos\theta = \frac{\langle -3, 8\rangle \cdot \langle -8, 3\rangle}{|\langle -3, 8\rangle||\langle -8, 3\rangle|}$$
$$\cos\theta = \frac{(-3)(-8)+8(3)}{\sqrt{(-3)^2+8^2}\sqrt{(-8)^2+3^2}}$$
$$\cos\theta = \frac{24+24}{\sqrt{73}\sqrt{73}}$$
$$\cos\theta = \frac{48}{73}$$
$$\theta = \cos^{-1}\frac{48}{73}$$
$$\theta \approx 48.9^\circ$$

180 – (65.6 + 48.9) = 65.6
The three angles are about 48.9°, 65.6°, and 65.6°.

Given u, |v| , and θ, the angle between u and v, find possible values of v. Round to the nearest hundredth.

60. **u** = $\langle 4, -2\rangle$, |**v**| = 10, 45°

SOLUTION:

Let $\mathbf{v} = \langle x, y\rangle$. Substitute the given values into the equation for the angle between to vectors..

$$\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos 45^\circ = \frac{\langle 4,-2\rangle\bullet\langle x,y\rangle}{|\langle 4,-2\rangle|\cdot 10}$$
$$\frac{\sqrt{2}}{2} = \frac{4x-2y}{\sqrt{20}\cdot 10}$$
$$\frac{\sqrt{2}}{2} = \frac{4x-2y}{20\sqrt{5}}$$
$$20\sqrt{5}\cdot\frac{\sqrt{2}}{2} = 4x-2y$$
$$10\sqrt{10} = 4x-2y$$

Solve $10\sqrt{10} = 4x - 2y$ for y.

$$10\sqrt{10} = 4x-2y$$
$$10\sqrt{10}-4x = -2y$$
$$\frac{10\sqrt{10}-4x}{-2} = y$$
$$-5\sqrt{10}+2x = y$$

Since $|\mathbf{v}| = 10$, $x^2 + y^2 = 100$. Substitute $y = -5\sqrt{10} + 2x$ into this equation and solve for x.

$$x^2+y^2 = 100$$
$$x^2+(-5\sqrt{10}+2x)^2 = 100$$
$$x^2+250-20\sqrt{10}x+4x^2 = 100$$
$$5x^2-20\sqrt{10}x+150 = 0$$

Use the Pythagorean Theorem.

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-20\sqrt{10})\pm\sqrt{(-20\sqrt{10})^2-4(5)(150)}}{2(5)}$$
$$= \frac{20\sqrt{10}\pm\sqrt{1000}}{10}$$
$$= \frac{20\sqrt{10}\pm 10\sqrt{10}}{10}$$
$$= 2\sqrt{10}\pm 1\sqrt{10}$$
$$\approx 9.49 \text{ or } 3.16$$

Find y when $x = 9.49$.

$$-5\sqrt{10}+2x = y$$
$$-5\sqrt{10}+2(9.49) = y$$
$$3.17 \approx y$$

Find y when $x = 3.16$.

$$-5\sqrt{10}+2x = y$$
$$-5\sqrt{10}+2(3.16) = y$$
$$-9.49 \approx y$$

$\langle 9.49, 3.17\rangle$ or $\langle 3.16, -9.49\rangle$

61. **u** = $\langle 3, 4\rangle$, |**v**| = $\sqrt{29}$, 121°

SOLUTION:

Let $\mathbf{v} = \langle x, y\rangle$. Substitute the given values into the equation for the angle between to vectors..

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos 121^\circ = \frac{\langle 3,4\rangle\cdot\langle x,y\rangle}{|\langle 3,4\rangle|\cdot\sqrt{29}}$$
$$\cos 121^\circ = \frac{3x+4y}{5\cdot\sqrt{29}}$$
$$5\sqrt{29}\cdot\cos 121^\circ = 3x+4y$$
$$-13.87 = 3x+4y$$
$$-13.87-3x = 4y$$
$$\frac{-13.87-3x}{4} = y$$

Since $|\mathbf{v}| = \sqrt{29}$, $x^2 + y^2 = 29$. Substitute $y = \frac{-13.87-3x}{4}$ into this equation and solve for x.

$$x^2 + y^2 = 29$$
$$x^2 + \left(\frac{-13.87-3x}{4}\right)^2 = 29$$
$$x^2 + \frac{192.38+83.22x+9x^2}{16} = 29$$
$$16x^2 + 192.38 + 83.22x + 9x^2 = 464$$
$$25x^2 + 83.22x - 271.62 = 0$$

Use the Pythagorean Theorem.

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(83.22)\pm\sqrt{(83.22)^2-4(25)(-271.62)}}{2(25)}$$
$$= \frac{-83.22\pm\sqrt{34{,}087.57}}{50}$$
$$\approx 2.03 \text{ or } -5.36$$

Find y when $x = 2.03$.

$$\frac{-13.87-3x}{4} = y$$
$$\frac{-13.87-3(2.03)}{4} = y$$
$$-4.99 \approx y$$

Find y when $x = -5.36$.

$$\frac{-13.87-3x}{4} = y$$
$$\frac{-13.87-3(-5.36)}{4} = y$$
$$0.55 \approx y$$

$\langle -5.36, 0.55\rangle$ or $\langle 2.03, -4.99\rangle$

62. $\mathbf{u} = \langle -1, -6\rangle$, $|\mathbf{v}| = 7$, 96°

SOLUTION:

Let $\mathbf{v} = \langle x, y\rangle$. Substitute the given values into the equation for the angle between to vectors..

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos 96^\circ = \frac{\langle -1,-6\rangle\cdot\langle x,y\rangle}{|\langle -1,-6\rangle|\cdot 7}$$
$$\cos 96^\circ = \frac{-x-6y}{\sqrt{37}\cdot 7}$$
$$7\sqrt{37}\cdot\cos 96^\circ = -x-6y$$
$$-4.45 = -x-6y$$
$$-4.45+x = -6y$$
$$\frac{-4.45+x}{-6} = y$$

Since $|\mathbf{v}| = 7$, $x^2 + y^2 = 49$. Substitute $y = \frac{-4.45+x}{-6}$ into this equation and solve for x.

$$x^2 + y^2 = 49$$
$$x^2 + \left(\frac{-4.45+x}{-6}\right)^2 = 49$$
$$x^2 + \frac{19.8-8.9x+x^2}{36} = 49$$
$$36x^2 + 19.8 - 8.9x + x^2 = 1764$$
$$37x^2 - 8.9x - 1744.2 = 0$$

Use the Pythagorean Theorem.

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-8.9)\pm\sqrt{(-8.9)^2-4(37)(-1744.2)}}{2(37)}$$
$$= \frac{8.9\pm\sqrt{258{,}220.81}}{74}$$
$$\approx 6.99 \text{ or } -6.75$$

Find y when $x = 6.99$.

$$\frac{-4.45+x}{-6} = y$$
$$\frac{-4.45+6.99}{-6} = y$$
$$-0.42 \approx y$$

Find y when $x = -6.75$.

$$\frac{-4.45+x}{-6}=y$$
$$\frac{-4.45+(-6.75)}{-6}=y$$
$$1.87\approx y$$

$\langle -6.75, 1.87\rangle$ or $\langle 6.99, -0.42\rangle$

63. $\mathbf{u} = \langle -2, 5\rangle$, $|\mathbf{v}| = 12$, 27°

SOLUTION:

Let $\mathbf{v}=\langle x,y\rangle$. Substitute the given values into the equation for the angle between to vectors..

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$
$$\cos 27^\circ=\frac{\langle -2,5\rangle\cdot\langle x,y\rangle}{|\langle -2,5\rangle|\cdot 12}$$
$$\cos 27^\circ=\frac{-2x+5y}{\sqrt{29}\cdot 12}$$
$$12\sqrt{29}\cdot\cos 27^\circ=-2x+5y$$
$$57.58=-2x+5y$$
$$57.58+2x=5y$$
$$\frac{57.58+2x}{5}=y$$

Since $|\mathbf{v}|=12$, $x^2+y^2=144$. Substitute $y=\frac{57.58+2x}{5}$ into this equation and solve for x.

$$x^2+y^2=144$$
$$x^2+\left(\frac{57.58+2x}{5}\right)^2=144$$
$$x^2+\frac{3315.46+230.32x+4x^2}{25}=144$$
$$25x^2+3315.46+230.32x+4x^2=3600$$
$$29x^2+230.32x-284.54=0$$

Use the Pythagorean Theorem.

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(230.32)\pm\sqrt{(230.32)^2-4(29)(-284.54)}}{2(29)}$$
$$=\frac{-230.32\pm\sqrt{86{,}053.94}}{58}$$
$$\approx 1.09 \text{ or } -9.03$$

Find y when $x = 1.09$.

$$\frac{57.58+2x}{5}=y$$
$$\frac{57.58+2(1.09)}{5}=y$$
$$11.95\approx y$$

Find y when $x = -9.03$.

$$\frac{57.58+2x}{5}=y$$
$$\frac{57.58+2(-9.03)}{5}=y$$
$$7.90\approx y$$

$\langle -9.03, 7.90\rangle$ or $\langle 1.09, 11.95\rangle$

64. **CARS** A car is stationary on a 9° incline. Assuming that the only forces acting on the car are gravity and the 275 Newton force applied by the brakes, about how much does the car weigh?

SOLUTION:

The car's weight is the force exerted due to gravity and can be represented by the force vector $\mathbf{F} = \langle 0, y \rangle$. The force required to keep the car from rolling is 275 newtons. Project **F** onto a unit vector **v** in the direction of the side of the incline to solve for the weight of the car.

Find a unit vector **v** in the direction of the side of the incline.

$$\begin{aligned}\mathbf{v} &= \langle |\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta \rangle \\ &= \langle 1(\cos 9^\circ), 1(\sin 9^\circ) \rangle \\ &= \langle \cos 9^\circ, \sin 9^\circ \rangle\end{aligned}$$

Find the projection of **F** onto the unit vector **v**, $\text{proj}_{\mathbf{v}}\mathbf{F}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\ &= (\mathbf{F}\cdot\mathbf{v})\mathbf{v} \\ &= (\langle 0, y \rangle \cdot \langle \cos 9^\circ, \sin 9^\circ \rangle)\mathbf{v} \\ &= (y\sin 9^\circ)\mathbf{v}\end{aligned}$$

The force required is (y sin 9°)**v**. Since **v** is a unit vector, the magnitude of the force required to keep the car stationary is (y sin 9°). Solve for y.

$$\begin{aligned}y\sin 9^\circ &= 275 \\ y &= \frac{275}{\sin 9^\circ} \\ y &\approx 1757.9\end{aligned}$$

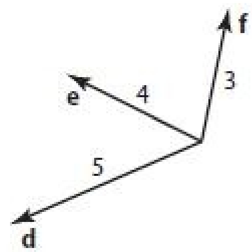
Since the force is given in units, the car's weight will be given in kilograms. Thus, the car weighs about 1757.9 kilograms.

65. **REASONING** Determine whether the statement below is *true* or *false*. Explain.
If |**d**|, |**e**|, and |**f**| form a Pythagorean triple, and the angles between **d** and **e** and between **e** and **f** are acute, then the angle between **d** and **f** must be a right angle. Explain your reasoning.

SOLUTION:

The statement is false. All three vectors may originate at the same point and not form a triangle at all. If so, the angle between **d** and **f** may be acute, right, or obtuse.

For example:



Drawing is not to scale.

|**d**|, |**e**|, and |**f**| form a Pythagorean triple, and the angles between **d** and **e** and between **e** and **f** are acute, but the angle between **d** and **f** is obtuse.

66. **ERROR ANALYSIS** Beng and Ethan are studying the properties of the dot product. Beng concludes that the dot product is associative because it is commutative; that is $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$. Ethan disagrees. Is either of them correct? Explain your reasoning.

SOLUTION:

Ethan is correct. The dot product of $\mathbf{u}\cdot\mathbf{v}$ will result in some value x. Since x is a scalar and not a vector, $x\cdot\mathbf{w}$ is undefined. So, $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$ is undefined.

67. **REASONING** Determine whether the statement below is *true* or *false*.
If **a** and **b** are both orthogonal to **v** in the plane, then **a** and **b** are parallel. Explain your reasoning.

SOLUTION:

The statement is false. If either **a** or **b** is a zero vector, it would be orthogonal to **v**, but could not be parallel to the other vector. Recall that the terms perpendicular and orthogonal have essentially the same meaning, except when **a** or **b** is the zero vector. The zero vector is orthogonal to any vector, but since it has no magnitude or direction, it cannot be perpendicular to a vector.

68. **CHALLENGE** If **u** and **v** are perpendicular, what is the projection of **u** onto **v**?

SOLUTION:

The equation for the projection of **u** onto **v** is $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$. If **u** and **v** are perpendicular, then they are orthogonal. So, $\mathbf{u}\cdot\mathbf{v} = 0$. Substitute $\mathbf{u}\cdot\mathbf{v} = 0$ into the equation for the projection of **u** onto **v**.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{0}{|\mathbf{v}|^2}\right)\mathbf{v}$$
$$\text{proj}_{\mathbf{v}}\mathbf{u} = 0$$

So, the projection of **u** onto **v** is 0.

69. **PROOF** Show that if the angle between vectors **u** and **v** is 90°, $\mathbf{u}\cdot\mathbf{v} = 0$ using the formula for the angle between two nonzero vectors.

SOLUTION:

$$\cos 90° = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|} \quad \text{Angle between } \mathbf{u} \text{ and } \mathbf{v} \text{ where } \theta = 90°.$$
$$0 = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|} \quad \cos 90° = 0.$$
$$0 = \mathbf{u}\cdot\mathbf{v} \quad \text{Multiply each side by } |\mathbf{u}|\cdot|\mathbf{v}|.$$

PROOF Prove each dot product property. Let $\mathbf{u} = \langle \mathbf{u_1}, \mathbf{u_2}\rangle$, $\mathbf{v} = \langle \mathbf{v_1}, \mathbf{v_2}\rangle$, and $\mathbf{w} = \langle \mathbf{w_1}, \mathbf{w_2}\rangle$.

70. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

SOLUTION:

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$$
$$\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle \stackrel{?}{=} \langle v_1, v_2\rangle \cdot \langle u_1, u_2\rangle$$
$$u_1v_1 + u_2v_2 = u_1v_1 + u_2v_2$$

71. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

SOLUTION:

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$$
$$\langle u_1, u_2\rangle \cdot (\langle v_1, v_2\rangle + \langle w_1, w_2\rangle) \stackrel{?}{=} \langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle + \langle u_1, u_2\rangle \cdot \langle w_1, w_2\rangle$$
$$\langle u_1, u_2\rangle \cdot \langle v_1 + w_1, v_2 + w_2\rangle \stackrel{?}{=} (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2)$$
$$u_1(v_1 + w_1) + u_2(v_2 + w_2) \stackrel{?}{=} u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2$$
$$u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 = u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2$$

72. $k(\mathbf{u}\cdot\mathbf{v}) = k\mathbf{u}\cdot\mathbf{v} = \mathbf{u}k\cdot\mathbf{v}$

SOLUTION:

$$k(\mathbf{u}\cdot\mathbf{v}) = k\mathbf{u}\cdot\mathbf{v} \qquad = \mathbf{u}\cdot k\mathbf{v}$$
$$k(\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle) \stackrel{?}{=} k\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle \qquad \stackrel{?}{=} \langle u_1, u_2\rangle \cdot k\langle v_1, v_2\rangle$$
$$k(u_1v_1 + u_2v_2) \stackrel{?}{=} \langle ku_1, ku_2\rangle \cdot \langle v_1, v_2\rangle \qquad \stackrel{?}{=} \langle u_1, u_2\rangle \cdot \langle kv_1, kv_2\rangle$$
$$ku_1v_1 + ku_2v_2 = ku_1v_1 + ku_2v_2 \qquad = ku_1v_1 + ku_2v_2$$

73. ***Writing in Math*** Explain how to find the dot product of two nonzero vectors.

SOLUTION:

Sample answer: For two nonzero vectors $\langle a, b\rangle$ and $\langle c, d\rangle$, the dot product $\langle a, b\rangle \cdot \langle c, d\rangle$ is the sum of the products of the *x*-coordinates and the *y*-coordinates, or $ac + bd$.

Find each of the following for $a = \langle 10, 1\rangle$, $b = \langle -5, 2.8\rangle$, **and** $c = \left\langle \frac{3}{4}, -9\right\rangle$.

74. **b** – **a** + 4**c**

SOLUTION:

$$\begin{aligned}\mathbf{b} - \mathbf{a} + 4\mathbf{c} &= \langle -5, 2.8\rangle - \langle 10, 1\rangle + (4)\left\langle \frac{3}{4}, -9\right\rangle \\ &= \langle -5, 2.8\rangle - \langle 10, 1\rangle + \langle 3, -36\rangle \\ &= \langle -15, 1.8\rangle + \langle 3, -36\rangle \\ &= \langle -12, -34.2\rangle\end{aligned}$$

75. **c** – 3**a** + **b**

SOLUTION:

$$\begin{aligned}\mathbf{c} - 3\mathbf{a} + \mathbf{b} &= \left\langle \frac{3}{4}, -9\right\rangle - (3)\langle 10, 1\rangle + \langle -5, 2.8\rangle \\ &= \left\langle \frac{3}{4}, -9\right\rangle - \langle 30, 3\rangle + \langle -5, 2.8\rangle \\ &= \left\langle -\frac{117}{4}, -12\right\rangle + \langle -5, 2.8\rangle \\ &= \left\langle -\frac{137}{4}, -9.2\right\rangle\end{aligned}$$

76. 2**a** – 4**b** + **c**

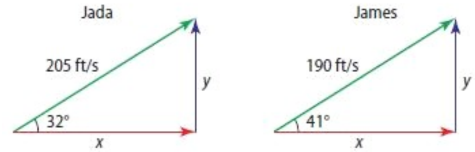
SOLUTION:

$$\begin{aligned}2\mathbf{a} - 4\mathbf{b} + \mathbf{c} &= (2)\langle 10, 1\rangle - (4)\langle -5, 2.8\rangle + \left\langle \frac{3}{4}, -9\right\rangle \\ &= \langle 20, 2\rangle - \langle -20, 11.2\rangle + \left\langle \frac{3}{4}, -9\right\rangle \\ &= \langle 40, -9.2\rangle + \left\langle \frac{3}{4}, -9\right\rangle \\ &= \left\langle \frac{163}{4}, -18.2\right\rangle\end{aligned}$$

77. **GOLF** Jada drives a golf ball with a velocity of 205 feet per second at an angle of 32° with the ground. On the same hole, James drives a golf ball with a velocity of 190 feet per second at an angle of 41°. Find the magnitudes of the horizontal and vertical components for each force.

SOLUTION:

Draw a diagram that shows the resolution of the force that Jada and James exert on the golf ball into its rectangular components.



The horizontal and vertical components of each force form a right triangle. Use the sine or cosine ratios to find the magnitude of each component.

For Jada:

$$\cos 32^\circ = \frac{x}{205} \qquad \sin 32^\circ = \frac{y}{205}$$
$$205\cos 32^\circ = x \qquad 205\sin 32^\circ = y$$
$$173.8 \approx x \qquad 108.6 \approx y$$

For James:

$$\cos 41^\circ = \frac{x}{190} \qquad \sin 41^\circ = \frac{y}{190}$$
$$190\cos 41^\circ = x \qquad 190\sin 41^\circ = y$$
$$143.4 \approx x \qquad 124.7 \approx y$$

So, the horizontal and vertical components for Jada are about 173.8 feet per second and about 108.6 feet per second. The horizontal and vertical components for James are about 143.4 feet per second and about 124.7 feet per second.

Graph the hyperbola given by each equation.

78. $\frac{(x-5)^2}{48}-\frac{y^2}{5}=1$

SOLUTION:

The equation is in standard form, where $h = 5$, $k = 0$, $a = \sqrt{48}$ or about 6.93, $b = \sqrt{5}$ or about 2.24, and $c = \sqrt{48+5}$ or about 7.28. In the standard form of the equation, the y−term is being subtracted. Therefore, the orientation of the hyperbola is horizontal.

center: $(h, k) = (5, 0)$

vertices: $(h \pm a, k) = (11.93, 0)$ and $(-1.93, 0)$

foci: $(h \pm c, k) = (12.28, 0)$ and $(-2.28, 0)$

asymptotes:

$$y-k = \pm\frac{b}{a}(x-h)$$

$$y-0 = \pm\frac{\sqrt{5}}{4\sqrt{3}}(x-5)$$

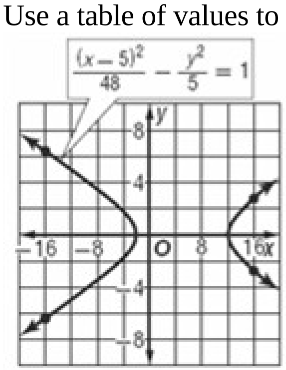
$$y = \pm\frac{\sqrt{15}}{12}(x-5)$$

$$y = \pm\left[\frac{\sqrt{15}}{12}x+\frac{5\sqrt{15}}{12}\right]$$

$$y = \frac{\sqrt{15}}{12}x+\frac{5\sqrt{15}}{12} \quad OR$$

$$y = -\frac{\sqrt{15}}{12}x-\frac{5\sqrt{15}}{12}$$

Use a table of values to sketch the hyperbola.



79. $\frac{x^2}{81}-\frac{y^2}{49}=1$

SOLUTION:

The equation is in standard form, where $h = 0$, $k = 0$, $a = 9$, $b = 7$, and $c = \sqrt{81+49}$ or about 11.4.

In the standard form of the equation, the y-term is being subtracted. Therefore, the orientation of the hyperbola is horizontal.

center: $(h, k) = (0, 0)$

vertices: $(h \pm a, k) = (9, 0)$ and $(-9, 0)$

foci: $(h \pm c, k) = (11.4, 0)$ and $(-11.4, 0)$

asymptotes:

$$y-k = \pm\frac{b}{a}(x-h)$$

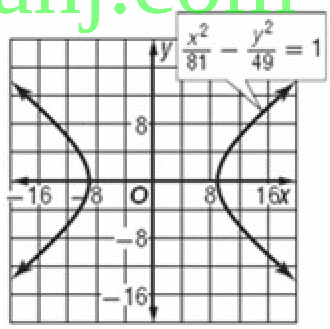
$$y-0 = \pm\frac{7}{9}(x-0)$$

$$y = \pm\frac{7}{9}x$$

$$y = \frac{7}{9}x \quad OR$$

$$y = -\frac{7}{9}x$$

Use a table of values to sketch the hyperbola.



80. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

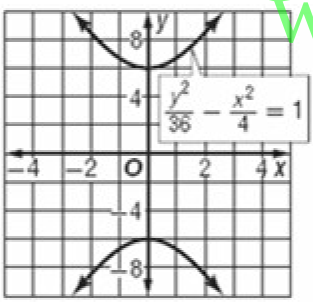
SOLUTION:

The equation is in standard form, where $h = 0$, $k = 0$, $a = 6$, $b = 2$, and $c = \sqrt{36+4}$ or about 6.32. In the standard form of the equation, the x-term is being subtracted. Therefore, the orientation of the hyperbola is vertical.

center: $(h, k) = (0, 0)$
vertices: $(h, k \pm a) = (0, 6)$ and $(0, -6)$
foci: $(h, k \pm c) = (0, 6.32)$ and $(0, -6.32)$
asymptotes:

$$\begin{aligned} y - k &= \pm\tfrac{a}{b}(x - h) \\ y - 0 &= \pm\tfrac{6}{2}(x - 9) \\ y &= \pm 3x \\ y &= 3x \quad OR \\ y &= -3x \end{aligned}$$

Use a table of values to sketch the hyperbola.



Find the exact value of each expression, if it exists.

81. $\arcsin\left(\sin\frac{\pi}{6}\right)$

SOLUTION:

First, find $\sin\frac{\pi}{6}$. On the unit circle, $\frac{\pi}{6}$ corresponds to $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. So, $\sin\frac{\pi}{6} = \frac{1}{2}$.

Next, find arcsin.

The inverse property applies, because $\frac{1}{2}$ is on the interval [–1, 1]. Therefore, $\arcsin\frac{1}{2} = \frac{\pi}{6}$, and $\arcsin\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$.

82. $\arctan\left(\tan\frac{1}{2}\right)$

SOLUTION:

The inverse property applies, because $\frac{1}{2}$ lies on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, $\arctan\left(\tan\frac{1}{2}\right) = \frac{1}{2}$.

83. $\sin\left(\cos^{-1}\frac{3}{4}\right)$

SOLUTION:

To simplify the expression, let $u = \cos^{-1}\left(\frac{3}{4}\right)$ so $\cos u = \frac{3}{4}$. Because the cosine function is positive in Quadrants I and IV, and the domain of the inverse cosine function is restricted to Quadrants I and II, u must lie in Quadrant I. Using the Pythagorean Theorem, you can find that the length of the opposite side is $\sqrt{7}$. $\sin u = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{7}}{4}$. So, $\sin\left(\cos^{-1}\frac{3}{4}\right) = \frac{\sqrt{7}}{4}$.

Solve each equation.

84. $\log_{12}(x^3 + 2) = \log_{12} 127$

SOLUTION:

$$\begin{aligned}\log_{12}(x^3+2) &= \log_{12}127\\ x^3+2 &= 127\\ x^3 &= 125\\ x &= 5\end{aligned}$$

85. $\log_2 x = \log_2 6 + \log_2 (x - 5)$

SOLUTION:

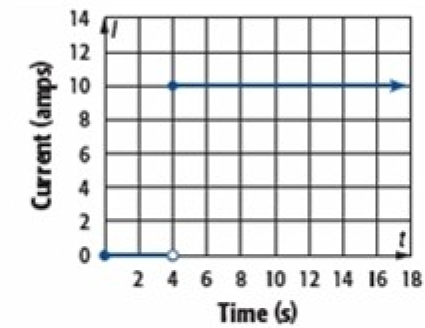
$$\begin{aligned}\log_2 x &= \log_2 6 + \log_2(x-5)\\ \log_2 x &= \log_2 6(x-5)\\ \log_2 x &= \log_2(6x-30)\\ x &= 6x-30\\ -5x &= -30\\ x &= 6\end{aligned}$$

86. $e^{5x-4} = 70$

SOLUTION:

$$\begin{aligned}e^{5x-4} &= 70\\ \ln e^{5x-4} &= \ln 70\\ 5x-4 &= \ln 70\\ 5x &= \ln 70 + 4\\ x &= \frac{\ln 70 + 4}{5}\\ x &\approx 1.65\end{aligned}$$

87. **ELECTRICITY** A simple electric circuit contains only a power supply and a resistor. When the power supply is off, there is no current in the circuit. When the power supply is turned on, the current almost instantly becomes a constant value. This situation can be modeled by a graph like the one shown. I represents current in amps, and t represents time in seconds.



a. At what t-value is this function discontinuous?

b. When was the power supply turned on?

c. If the person who turned on the power supply left and came back hours later, what would he or she measure the current in the circuit to be?

SOLUTION:

a. There appears to be a jump discontinuity at $t = 4$. Therefore, the graph is discontinuous at $t = 4$.

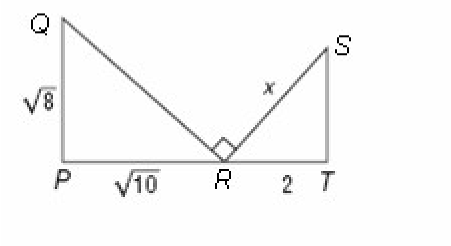
b. The current is 0 amps for $0 \le t < 4$ and 10 amps for $4 \le t$. Therefore, it appears that the power supply was turned on at $t = 4$.

c. From the graph, it appears $\lim_{t \to \infty} I(t) = 10$.

Therefore, the current in the circuit would be 10 amps.

8-3 Dot Products and Vector Projections

88. **SAT/ACT** In the figure below, $\Delta PQR \sim \Delta TRS$. What is the value of x?



A $\sqrt{2}$
B $\sqrt{5}$
C 3
D $3\sqrt{2}$
E 6

SOLUTION:

Since $\Delta PQR \sim \Delta TRS$, then $\frac{PQ}{TR} = \frac{QR}{RS}$. Use the Pythagorean Theorem to solve for QR.

$$(PR)^2 + (PQ)^2 = (QR)^2$$
$$\left(\sqrt{10}\right)^2 + \left(\sqrt{8}\right)^2 = (QR)^2$$
$$18 = (QR)^2$$
$$3\sqrt{2} = QR$$

Substitute PQ, TR, and QR into $\frac{PQ}{TR} = \frac{QR}{RS}$ and solve for RS.

$$\frac{\sqrt{8}}{2} = \frac{3\sqrt{2}}{RS}$$
$$RS\left(\sqrt{8}\right) = 2\left(3\sqrt{2}\right)$$
$$RS = \frac{2\left(3\sqrt{2}\right)}{\left(\sqrt{8}\right)}$$
$$RS = 3$$

So, $x = 3$.
The correct answer is C.

89. **REVIEW** Consider $C(-9, 2)$ and $D(-4, -3)$. Which of the following is the component form and magnitude of $\overrightarrow{CD}$?

F $\langle 5, -5\rangle$, $5\sqrt{2}$
G $\langle 5, -5\rangle$, $6\sqrt{2}$
H $\langle 6, -5\rangle$, $5\sqrt{2}$
J $\langle 6, -6\rangle$, $6\sqrt{2}$

SOLUTION:

First, find the component form.

$$\overrightarrow{CD} = \langle x_2 - x_1, y_2 - y_1\rangle$$
$$= \langle -4 - (-9), -3 - 2\rangle$$
$$= \langle 5, -5\rangle$$

Next, find the magnitude. Substitute $x_2 - x_1 = 5$ and $y_2 - y_1 = -5$ into the formula for the magnitude of a vector in the coordinate plane.

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{5^2 + (-5)^2}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2}$$

The correct answer is F.

90. A snow sled is pulled by exerting a force of 25 pounds on a rope that makes a 20° angle with the horizontal, as shown in the figure. What is the approximate work done in pulling the sled 50 feet?

A 428 foot-pounds
B 1093 foot-pounds
C 1175 foot-pounds
D 1250 foot-pounds

SOLUTION:

Use the projection formula for work. The magnitude of the projection of **F** onto $\overrightarrow{AB}$ is $|\mathbf{F}|\cos\theta = 25\cos 20°$. The magnitude of the directed distance $\overrightarrow{AB}$ is 50.

$$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}||\overrightarrow{AB}|$$
$$= (25\cos 20°)(50)$$
$$\approx 1174.6$$

Therefore, the approximate work done in pulling the sled is about 1174.6 foot-pounds.
The correct answer is C.

91. **REVIEW** If $\mathbf{s} = \langle 4, -3\rangle$, $\mathbf{t} = \langle -6, 2\rangle$, which of the following represents $\mathbf{t} - 2\mathbf{s}$?
F $\langle 14, 8\rangle$
G $\langle 14, 6\rangle$
H $\langle -14, 8\rangle$
J $\langle -14, -8\rangle$

SOLUTION:

$$\mathbf{t} - 2\mathbf{s} = \mathbf{t} + (-2)\mathbf{s}$$
$$= \langle -6, 2\rangle + (-2)\langle 4, -3\rangle$$
$$= \langle -6, 2\rangle + \langle -8, 6\rangle$$
$$= \langle -14, 8\rangle$$

The correct answer is H.