Chapter 3

Applications of Differentiation

3.1 Linear Approximations and Newtons Method

1. (a) $f(x) = \sqrt{x}, x_0 = 1$
$f(x_0) = f(1) = \sqrt{1} = 1$
$f'(x) = \frac{1}{2}x^{-1/2}$
$f'(x_0) = f'(1) = \frac{1}{2}$
So,
$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 1 + \frac{1}{2}(x-1) = \frac{1}{2} + \frac{1}{2}x$$

(b) Using the approximation $L(x)$ to estimate $\sqrt{1.2}$, we get $\sqrt{1.2} = f(1.2) \approx L(1.2) = \frac{1}{2} + \frac{1}{2}(1.2) = 1.1$

2. (a) $f(x_0) = f(0) = 1$ and
$f'(x) = \frac{1}{3}(x+1)^{-2/3}$
So, $f'(0) = \frac{1}{3}$
The Linear approximation is,
$L(x) = 1 + \frac{1}{3}(x-0) = 1 + \frac{1}{3}x$

(b) Using the approximation $L(x)$ to estimate $\sqrt[3]{1.2}$, we get $\sqrt[3]{1.2} = f(0.2) \approx L(0.2) = 1 + \frac{1}{3}(0.2) = 1.066$

3. (a) $f(x) = \sqrt{2x+9}, x_0 = 0$
$f(x_0) = f(0) = \sqrt{2 \cdot 0 + 9} = 3$
$$f'(x) = \frac{1}{2}(2x+9)^{-1/2} \cdot 2 = (2x+9)^{-1/2}$$
$f'(x_0) = f'(0) = (2 \cdot 0 + 9)^{-1/2} = \frac{1}{3}$
So,
$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 3 + \frac{1}{3}(x-0) = 3 + \frac{1}{3}x$$

(b) Using the approximation $L(x)$ to estimate $\sqrt{8.8}$, we get $\sqrt{8.8} = f(-0.1) \approx L(-0.1) = 3 + \frac{1}{3}(-0.1) = 3 - 0.033 = 2.967$

4. (a) $f(x) = \frac{2}{x}, x_0 = 1$
$f(x_0) = f(1) = 2$
$f'(x) = -\frac{2}{x^2}$ and so $f'(1) = -2$
The linear approximation is
$L(x) = 2 + (-2)(x-1)$

(b) Using the approximation $L(x)$ to estimate $\frac{2}{0.99}$, we get $\frac{2}{0.99} = f(0.99) \approx L(0.99) = 2 + (-2)(0.99 - 1) = 2.02$

5. (a) $f(x) = \sin 3x, x_0 = 0$
$f(x_0) = \sin(3 \cdot 0) = 0$
$f'(x) = 3\cos 3x$
$f'(x_0) = f'(0) = 3\cos(3 \cdot 0) = 3$
So,
$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 0 + 3(x-0) = 3x$$

(b) Using the approximation $L(x)$ to estimate $\sin(0.3)$, we get $\sin(0.3) = f(0.1) \approx L(0.1) = 3(0.1) = 0.3$

6. (a) $f(x) = \sin x, x_0 = \pi$
$f(x_0) = \sin \pi = 0$
$f'(x) = \cos x$
$f'(x_0) = f'(\pi) = \cos \pi = -1$
The linear approximation is,
$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 0 + (-1)(x - \pi) = \pi - x$$

(b) Using the approximation $L(x)$ to estimate $\sin(3.0)$, we get $\sin(3.0) = f(3.0) \approx L(3.0) = \pi - 3.0$

7. (a) $f(x) = \sqrt[4]{16+x}, x_0 = 0$
$f(0) = \sqrt[4]{16+0} = 2$
$f'(x) = \frac{1}{4}(16+x)^{-3/4}$
$f'(0) = \frac{1}{4}(16+0)^{-3/4} = \frac{1}{32}$

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= 2 + \frac{1}{32}x \\ &= 2 + \frac{1}{32}(0.04) = 2.00125 \end{aligned}$$

(b) $L(0.08) = 2 + \frac{1}{32}(0.08) = 2.0025$

(c) $L(0.16) = 2 + \frac{1}{32}(0.16) = 2.005$

8. (a) $f(x) = \sin x,\ x_0 = 0$
$f(0) = 0$
$f'(x) = \cos x$
$f'(0) = \cos 0 = 1$
$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= 0 + 1 \cdot x \end{aligned}$$
$L(0.1) = 0.1$

(b) $f(x) = \sin x,\ x_0 = \frac{\pi}{3}$
$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$
$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$
$L(1) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(1 - \frac{\pi}{3}\right) \approx 0.842$

(c) $f(x) = \sin x,\ x_0 = \frac{2\pi}{3}$
$f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$f'\left(\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$
$$\begin{aligned} L(x) &= f\left(\frac{2\pi}{3}\right) + f'\left(\frac{2\pi}{3}\right)\left(x - \frac{2\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{2\pi}{3}\right) \end{aligned}$$
$L\left(\frac{9}{4}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{9}{4} - \frac{2\pi}{3}\right) \approx 0.788$

9. (a) $L(x) = f(20) + \frac{18-14}{20-30}(x-20)$
$$\begin{aligned} L(24) &\approx 18 - \frac{4}{10}(24-20) \\ &= 18 - 0.4(4) \\ &= 16.4 \text{ games} \end{aligned}$$

(b) $L(x) = f(40) + \frac{14-12}{30-40}(x-40)$
$$\begin{aligned} f(36) &\approx 12 - \frac{2}{10}(36-40) \\ &= 12 - 0.2(-4) \\ &= 12.8 \text{ games} \end{aligned}$$

10. (a) $L(x) = f(80) + \frac{120-84}{80-60}(x-80)$
$$\begin{aligned} L(72) &= 120 + \frac{36}{20}(72-80) \\ &= 120 + 1.8(-8) \\ &= 105.6 \text{ cans} \end{aligned}$$

(b) $L(x) = f(100) + \frac{168-120}{100-80}(x-100)$
$$\begin{aligned} L(94) &= 168 - \frac{48}{20}(94-100) \\ &= 168 - 2.4(-6) \\ &= 182.4 \text{ cans} \end{aligned}$$

11. (a) $L(x) = f(200) + \frac{142-128}{220-200}(x-200)$
$$\begin{aligned} L(208) &= 128 + \frac{14}{20}(208-200) \\ &= 128 + 0.7(8) = 133.6 \end{aligned}$$

(b) $L(x) = f(240) + \frac{142-136}{220-240}(x-240)$
$$\begin{aligned} L(232) &= 136 - \frac{6}{20}(232-240) \\ &= 136 - 0.3(-8) = 138.4 \end{aligned}$$

12. (a) $L(x) = f(10) + \frac{14-8}{10-5}(x-10)$
$L(8) = 14 + \frac{6}{5}(-2) = 11.6$

(b) $L(x) = f(10) + \frac{14-8}{10-5}(x-10)$
$L(12) = 14 + \frac{6}{5}(2) = 16.4$

13. $f(x) = x^3 + 3x^2 - 1 = 0,\ x_0 = 1$
$f'(x) = 3x^2 + 6x$

(a)
$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{1^3 + 3 \cdot 1^2 - 1}{3 \cdot 1^2 + 6 \cdot 1} \\ &= 1 - \frac{3}{9} = \frac{2}{3} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= \frac{2}{3} - \frac{\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^2 - 1}{3\left(\frac{2}{3}\right)^2 + 6\left(\frac{2}{3}\right)} \\ &= \frac{79}{144} \approx 0.5486 \end{aligned}$$

(b) 0.53209

14. $f(x) = x^3 + 4x^2 - x - 1,\ x_0 = -1$
$f'(x) = 3x^2 + 8x - 1$

(a) $$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= -1 - \frac{3}{-6} = -\frac{1}{2} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= -\frac{1}{2} - \frac{0.375}{-4.25} = -0.4117647 \end{aligned}$$

(b) The root is $x \approx -0.4064206546$.

15. $f(x) = x^4 - 3x^2 + 1 = 0$, $x_0 = 1$
$f'(x) = 4x^3 - 6x$

(a) $$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \left(\frac{1^4 - 3 \cdot 1^2 + 1}{4 \cdot 1^3 - 6 \cdot 1}\right) = \frac{1}{2} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= \frac{1}{2} - \left(\frac{\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)^2 + 1}{4\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)}\right) \\ &= \frac{5}{8} \end{aligned}$$

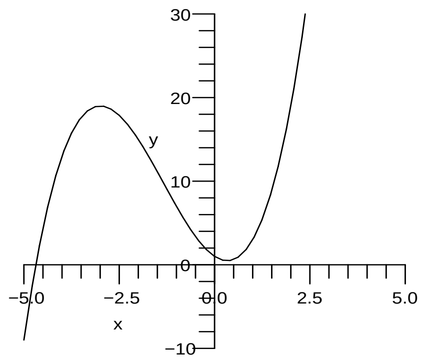
(b) 0.61803

16. $f(x) = x^4 - 3x^2 + 1$, $x_0 = -1$
$f'(x) = 4x^3 - 6x$

(a) $$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= -1 - \frac{-1}{2} = -\frac{1}{2} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= -\frac{1}{2} - \frac{0.3125}{2.5} = -0.625 \end{aligned}$$

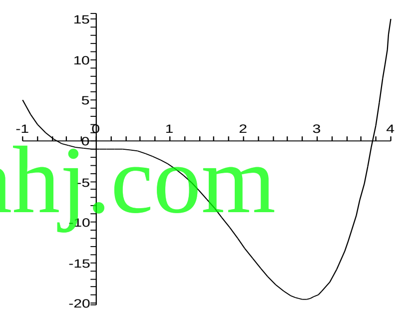
(b) The root is $x \approx -0.6180339887$.

17. Use $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ with
$f(x) = x^3 + 4x^2 - 3x + 1$, and
$f'(x) = 3x^2 + 8x - 3$



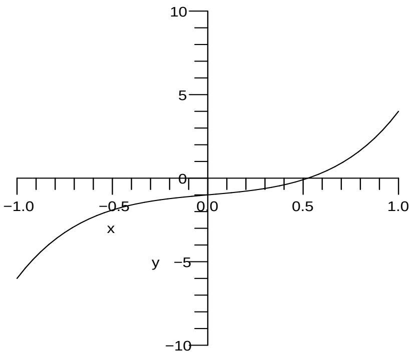
Start with $x_0 = -5$ to find the root near -5:
$x_1 = -4.718750$, $x_2 = -4.686202$,
$x_3 = -4.6857796$, $x_4 = -4.6857795$

18. From the graph, we see two roots:



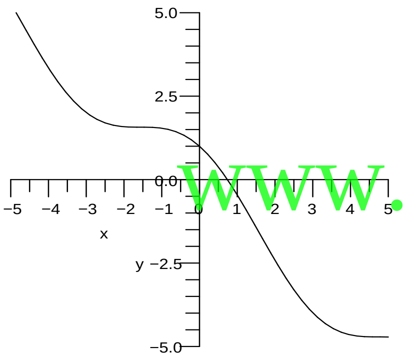
Use $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ with
$f(x) = x^4 - 4x^3 + x^2 - 1$, and
$f'(x) = 4x^3 - 12x^2 + 2x$
Start with $x_0 = 4$ to find the root below 4:
$x_1 = 3.791666667$, $x_2 = 3.753630030$, $x_3 = 3.7524339$, $x_4 = 3.752432297$
Start with $x = -1$ to find the root just above -1:
$x_1 = -0.7222222222$,
$x_2 = -0.5810217936$,
$x_3 = -0.5416512863$,
$x_4 = -0.5387668233$,
$x_5 = -0.5387519962$

19. Use $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ with
$f(x) = x^5 + 3x^3 + x - 1$, and
$f'(x) = 5x^4 + 9x^2 + 1$



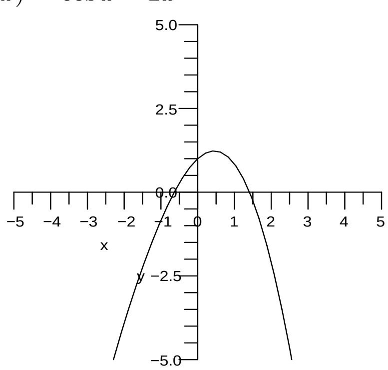
Start with $x_0 = 0.5$ to find the root near 0.5:
$x_1 = 0.526316$, $x_2 = 0.525262$,
$x_3 = 0.525261$, $x_4 = 0.525261$

20. Use $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ with
$f(x) = \cos x - x$, and
$f'(x) = -\sin x - 1$



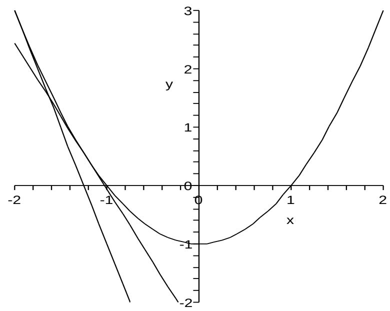
Start with $x_0 = 1$ to find the root near 1:
$x_1 = 0.750364$, $x_2 = 0.739113$,
$x_3 = 0.739085$, $x_4 = 0.739085$

21. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with
$f(x) = \sin x - x^2 + 1$, and
$f'(x) = \cos x - 2x$

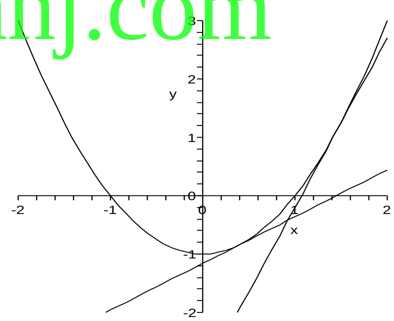


Start with $x_0 = -0.5$ to find the root near -0.5:
$x_1 = -0.644108$, $x_2 = -0.636751$
$x_3 = -0.636733$, $x_4 = -0.636733$
Start with $x_0 = 1.5$ to find the root near 1.5:
$x_1 = 1.413799$, $x_2 = 1.409634$
$x_3 = 1.409624$, $x_4 = 1.409624$

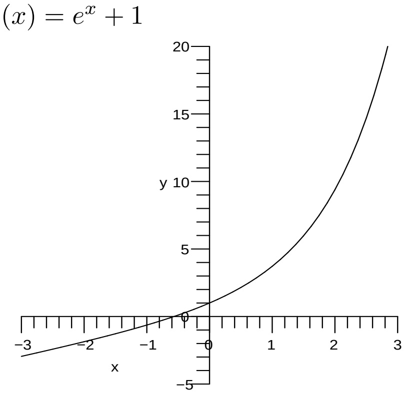
22. Use $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with
$f(x) = \cos x^2 - x$, and
$f'(x) = 2x \sin x^2 - 1$



Start with $x_0 = 1$ to find the root between 0 and 1:
$x_1 = 0.8286590991$, $x_2 = 0.8016918647$,
$x_3 = 0.8010710854$, $x_4 = 0.8010707652$



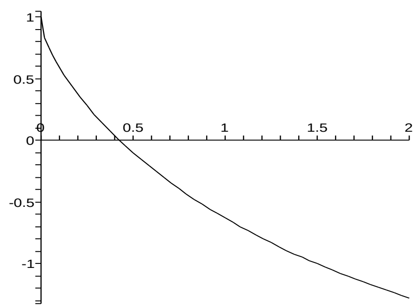
23. Use $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ with
$f(x) = e^x + x$, and
$f'(x) = e^x + 1$



Start with $x_0 = -0.5$ to find the root between

0 and -1:
$x_1 = -0.566311$, $x_2 = -0.567143$
$x_3 = -0.567143$, $x_4 = -0.567143$

24. Use $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ with
$f(x) = e^{-x} - \sqrt{x}$, and
$f'(x) = -e^{-x} - \dfrac{1}{2\sqrt{x}}$



Start with $x_0 = 0.5$ to find the root close to 0.5:
$x_1 = 0.4234369253$, $x_2 = 0.4262982542$,
$x_3 = 0.4263027510$

25. $f(x) = x^2 - 11$; $x_0 = 3$; $\sqrt{11} \approx 3.316625$

26. Newton's method for $\sqrt{x}$ near $x = 23$ is $x_{n+1} = \frac{1}{2}(x_n + 23/x_n)$. Start with $x_0 = 5$ to get: $x_1 = 4.8$, $x_2 = 4.7958333$, and $x_3 = 4.7958315$.

27. $f(x) = x^3 - 11$; $x_0 = 2$; $\sqrt[3]{11} \approx 2.22398$

28. Newton's method for $\sqrt[3]{x}$ near $x = 23$ is $x_{n+1} = \frac{1}{3}(2x_n + 23/x_n^2)$. Start with $x_0 = 3$ to get:
$x_1 = 2.851851851$, $x_2 = 2.843889316$, and
$x_3 = 2.884386698$

29. $f(x) = x^{4.4} - 24$; $x_0 = 2$; $\sqrt[4.4]{24} \approx 2.059133$

30. Newton's method for $\sqrt[4.6]{x}$ near $x = 24$ is $x_{n+1} = \frac{1}{4.6}(3.6x_n + 24/x_n^{3.6})$. Start with $x_0 = 2$ to get:
$x_1 = 1.995417100$, $x_2 = 1.995473305$, and
$x_3 = 1.995473304$

31. $f(x) = 4x^3 - 7x^2 + 1 = 0$, $x_0 = 0$
$f'(x) = 12x^2 - 14x$
$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0 - \dfrac{1}{0}$
The method fails because $f'(x_0) = 0$. Roots are 0.3454, 0.4362, 1.659.

32. Newton's method fails because $f' = 0$. As long as the sequence avoids $x_n = 0$ and $x_n = \dfrac{7}{6}$ (the zeros of f'), Newton's method will succeed. Which root is found depends on the starting place.

33. $f(x) = x^2 + 1$, $x_0 = 0$
$f'(x) = 2x$
$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0 - \dfrac{1}{0}$
The method fails because $f'(x_0) = 0$. There are no roots.

34. Newton's method fails because the function does not have a root!

35. $f(x) = \dfrac{4x^2 - 8x + 1}{4x^2 - 3x - 7} = 0$, $x_0 = -1$
Note: $f(x_0) = f(-1)$ is undefined, so Newton's Method fails because x_0 is not in the domain of f. Notice that $f(x) = 0$ only when $4x^2 - 8x + 1 = 0$. So using Newton's Method on $g(x) = 4x^2 - 8x + 1$ with $x_0 = -1$ leads to $x \approx .1339$. The other root is $x \approx 1.8660$.

36. The slope tends to infinity at the zero. For every starting point, the sequence does not converge.

37. (a) With $x_0 = 1.2$,
$x_1 = 0.800000000$,
$x_2 = 0.950000000$,
$x_3 = 0.995652174$,
$x_4 = 0.999962680$,
$x_5 = 0.999999997$,
$x_6 = 1.000000000$,
$x_7 = 1.000000000$

(b) With $x_0 = 2.2$,
$x_0 = 2.200000$, $x_1 = 2.107692$,
$x_2 = 2.056342$, $x_3 = 2.028903$,
$x_4 = 2.014652$, $x_5 = 2.007378$,
$x_6 = 2.003703$, $x_7 = 2.001855$,
$x_8 = 2.000928$, $x_9 = 2.000464$,
$x_{10} = 2.000232$, $x_{11} = 2.000116$,
$x_{12} = 2.000058$, $x_{13} = 2.000029$,
$x_{14} = 2.000015$, $x_{15} = 2.000007$,
$x_{16} = 2.000004$, $x_{17} = 2.000002$,
$x_{18} = 2.000001$, $x_{19} = 2.000000$,
$x_{20} = 2.000000$
The convergence is much faster with $x_0 = 1.2$.

38. Starting with $x_0 = 0.2$ we get a sequence that converges to 0 very slowly. (The 20th iteration is still not accurate past 7 decimal places). Starting with $x_0 = 3$ we get a sequence that quickly converges to π. (The third iteration is already accurate to 10 decimal places!)

39. (a) With $x_0 = -1.1$
$x_1 = -1.0507937$,
$x_2 = -1.0256065$,
$x_3 = -1.0128572$,
$x_4 = -1.0064423$,
$x_5 = -1.0032246$,
$x_6 = -1.0016132$,
$x_7 = -1.0008068$,
$x_8 = -1.0004035$,
$x_9 = -1.0002017$,
$x_{10} = -1.0001009$,
$x_{11} = -1.0000504$,
$x_{12} = -1.0000252$,
$x_{13} = -1.0000126$,
$x_{14} = -1.0000063$,
$x_{15} = -1.0000032$,
$x_{16} = -1.0000016$,
$x_{17} = -1.0000008$,
$x_{18} = -1.0000004$,
$x_{19} = -1.0000002$,
$x_{20} = -1.0000001$,
$x_{21} = -1.0000000$,
$x_{22} = -1.0000000$

(b) With $x_0 = 2.1$
$x_0 = 2.100000000$,
$x_1 = 2.006060606$,
$x_2 = 2.000024340$,
$x_3 = 2.000000000$,
$x_4 = 2.000000000$
The rate of convergence in (a) is slower than the rate of convergence in (b).

40. From exercise 37, $f(x) = (x-1)(x-2)^2$. Newton's method converges slowly near the double root. From exercise 39, $f(x) = (x-2)(x+1)^2$. Newton's method again converges slowly near the double root. In exercise 38, Newton's method converges slowly near 0, which is a zero of both x and $\sin x$ but converges quickly near π, which is a zero only of $\sin x$.

41. $f(x) = \tan x$, $f(0) = \tan 0 = 0$
$f'(x) = \sec^2 x$, $f'(0) = \sec^2 0 = 1$
$L(x) = f(0) + f'(0)(x-0)$ $L(0.01) = 0.01$
$= 0 + 1(x-0) = x$
$f(0.01) = \tan 0.01 \approx 0.0100003$
$L(0.1) = 0.1$
$f(0.1) = \tan(0.1) \approx 0.1003$
$L(1) = 1$
$f(1) = \tan 1 \approx 1.557$

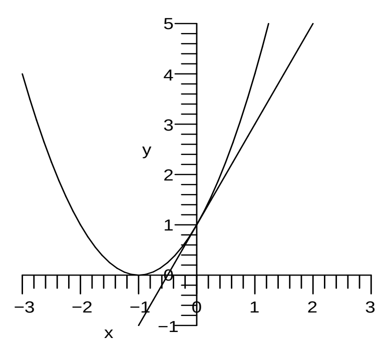
42. The linear approximation for $\sqrt{1+x}$ at $x = 0$ is $L(x) = 1 + \frac{1}{2}x$. The error when $x = 0.01$ is 0.0000124, when $x = 0.1$ is 0.00119, and when $x = 1$ is 0.0858.

43. $f(x) = \sqrt{4+x}$
$f(0) = \sqrt{4+0} = 2$
$f'(x) = \frac{1}{2}(4+x)^{-1/2}$
$f'(0) = \frac{1}{2}(4+0)^{-1/2} = \frac{1}{4}$
$L(x) = f(0) + f'(0)(x-0) = 2 + \frac{1}{4}x$
$L(0.01) = 2 + \frac{1}{4}(0.01) = 2.0025$
$f(0.01) = \sqrt{4+0.01} \approx 2.002498$
$L(0.1) = 2 + \frac{1}{4}(0.1) = 2.025$
$f(0.1) = \sqrt{4+0.1} \approx 2.0248$
$L(1) = 2 + \frac{1}{4}(1) = 2.25$
$f(1) = \sqrt{4+1} \approx 2.2361$

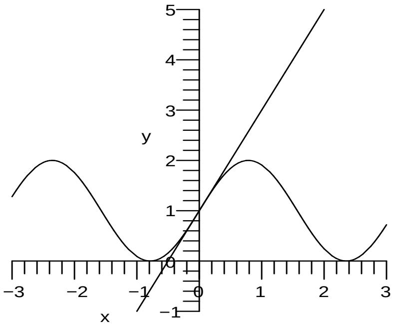
44. The linear approximation for e^x at $x = 0$ is $L(x) = 1 + x$. The error when $x = 0.01$ is 0.0000502, when $x = 0.1$ is 0.00517, and when $x = 1$ is 0.718.

45. (a) $f(0) = g(0) = h(0) = 1$, so all pass through the point (0, 1).
$f'(0) = 2(0+1) = 2$,
$g'(0) = 2\cos(2 \cdot 0) = 2$, and
$h'(0) = 2e^{2 \cdot 0} = 2$,
so all have slope 2 at $x = 0$.
The linear approximation at $x = 0$ for all three functions is $L(x) = 1 + 2x$.

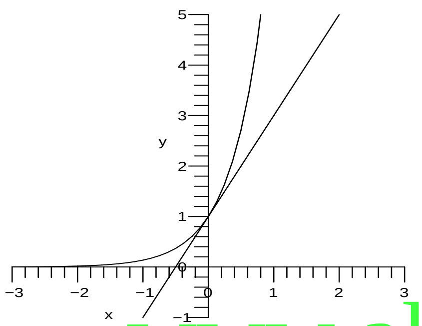
(b) Graph of $f(x) = (x+1)^2$:



Graph of $f(x) = 1 + \sin(2x)$:

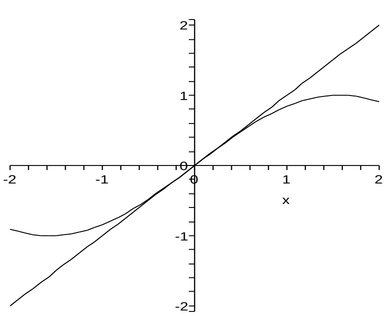


Graph of $f(x) = e^{2x}$:

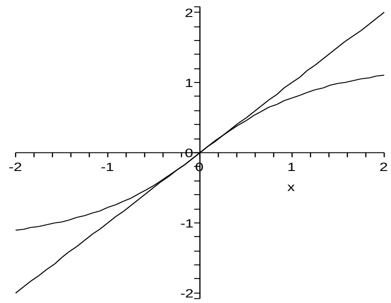


46. (a) $f(0) = g(0) = h(0) = 0$, so all pass through the point (0, 0).
$f'(0) = \cos 0 = 1$,
$g'(0) = \dfrac{1}{1+0^2} = 1$, and
$h'(0) = \cosh 0 = 1$,
so all have slope 1 at $x = 0$.
The linear approximation at $x = 0$ for all three functions is $L(x) = x$.

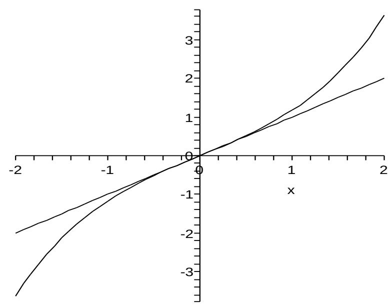
(b) Graph of $f(x) = \sin x$:



Graph of $g(x) = \tan^{-1} x$:



Graph of $h(x) = \sinh x$:



$\sin x$ is the closest fit, but $\sinh x$ is close.

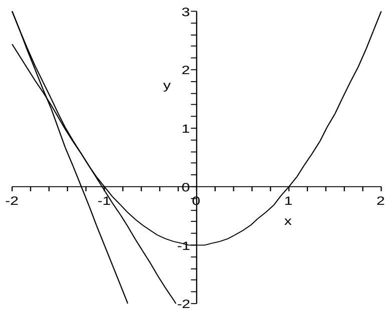
47. (a) $\sqrt[4]{16.04} = 2.0012488$
$L(0.04) = 2.00125$
$|2.0012488 - 2.00125| = .00000117$

(b) $\sqrt[4]{16.08} = 2.0024953$
$L(.08) = 2.0025$
$|2.0024953 - 2.0025| = .00000467$

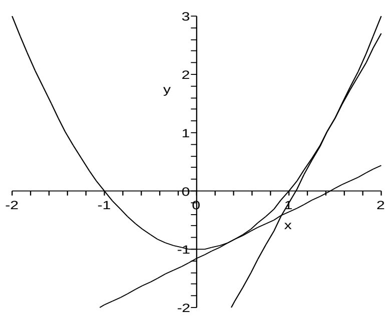
(c) $\sqrt[4]{16.16} = 2.0049814$
$L(.16) = 2.005$
$|2.0049814 - 2.005| = .0000186$

48. If you graph $|\tan x - x|$, you see that the difference is less than .01 on the interval $-.306 < x < .306$ (In fact, a slightly larger interval would work as well).

49. The first tangent line intersects the x-axis at a point a little to the right of 1. So x_1 is about 1.25 (very roughly). The second tangent line intersects the x-axis at a point between 1 and x_1, so x_2 is about 1.1 (very roughly). Newton's Method will converge to the zero at $x = 1$.
Starting with $x_0 = -2$, Newton's method converges to $x = -1$.



Starting with $x_0 = 0.4$, Newton's method converges to $x = 1$.



50. It wouldn't work because $f'(0) = 0$. $x_0 = 0.2$ works better as an initial guess. After jumping to $x_1 = 2.55$, the sequence rapidly decreases toward $x = 1$. Starting with $x_0 = 10$, it takes several steps to get to 2.5, on the way to $x = 1$.

51.
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left(\frac{x_n^2 - c}{2x_n}\right) = x_n - \frac{x_n^2}{2x_n} + \frac{c}{2x_n} = \frac{x_n}{2} + \frac{c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$
If $x_0 < \sqrt{a}$, then $a/x_0 > \sqrt{a}$, so $x_0 < \sqrt{a} < a/x_0$.

52. The root of $x^n - c$ is $\sqrt[n]{c}$, so Newton's method approximates this number.
Newton's method gives
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^n - c}{nx_i^{n-1}} = \frac{1}{n}(nx_i - x_i + cx_i^{1-n}),$$
as desired.

53. (a) $f(x) = x^2 - x - 1$
$f'(x) = 2x - 1$
At $x_0 = \dfrac{3}{2}$
$$f(x_0) = \left(\frac{3}{2}\right)^2 - \frac{3}{2} - 1 = -\frac{1}{4}$$
and
$$f'(x_0) = 2\left(\frac{3}{2}\right) - 1 = 2$$
By Newton's formula,
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{3}{2} - \frac{-\frac{1}{4}}{2} = \frac{13}{8}$$

(b) $f(x) = x^2 - x - 1$
$f'(x) = 2x - 1$
At $x_0 = \frac{5}{3}$
$$f(x_0) = \left(\frac{5}{3}\right)^2 - \frac{5}{3} - 1 = \frac{1}{9}$$
and
$$f'(x_0) = 2\left(\frac{5}{3}\right) - 1 = \frac{7}{3}$$
By Newton's formula,
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{5}{3} - \frac{\frac{1}{9}}{\frac{7}{3}} = \frac{5}{3} - \frac{1}{21} = \frac{34}{21}$$

(c) $f(x) = x^2 - x - 1$
$f'(x) = 2x - 1$
At $x_0 = \frac{8}{5}$
$$f(x_0) = \left(\frac{8}{5}\right)^2 - \frac{8}{5} - 1 = -\frac{1}{25}$$
and
$$f'(x_0) = 2\left(\frac{8}{5}\right) - 1 = \frac{11}{5}$$
By Newton's formula,
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{8}{5} - \frac{-\frac{1}{25}}{\frac{11}{5}} = \frac{8}{5} + \frac{1}{55} = \frac{89}{55}$$

(d) From part (a),
since$x_0 = \dfrac{F_4}{F_3}$, hence $x_1 = \dfrac{F_7}{F_6}$.
From part (b),
since $x_0 = \dfrac{F_5}{F_4}$ hence $x_1 = \dfrac{F_9}{F_8}$.
From part (c),
since $x_0 = \dfrac{F_6}{F_5}$ hence $x_1 = \dfrac{F_{11}}{F_{10}}$.
Thus in general if $x_0 = \dfrac{F_{n+1}}{F_n}$, then $x_1 = \dfrac{F_{2n+1}}{F_{2n}}$ implies $m = 2n + 1$ and $k = 2n$

(e) Given $x_0 = \dfrac{3}{2}$, then $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n}$ will be the zero of the function $f(x) = x^2 - x - 1$ which is 1.618034. Therefore, $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n} = 1.618034$

54. The general form of function $f(x)$ is,
$f_n(x) = \frac{1}{5}\left(2^{n+2}x - 3\right)$ for $\frac{1}{2^n} < x < \frac{1}{2^{n-1}}$.
Hence
$f'(x) = f_n{}'(x) = \frac{2^{n+2}}{5}$ for $\frac{1}{2^n} < x < \frac{1}{2^{n-1}}$.
By Newton's method,
$$x_1 = \frac{3}{4} - \frac{f\left(\frac{3}{4}\right)}{f'\left(\frac{3}{4}\right)} = \frac{3}{4} - \frac{f_1\left(\frac{3}{4}\right)}{f_1{}'\left(\frac{3}{4}\right)}$$
$$= \frac{3}{4} - \frac{(3/5\ )}{(8/5\ )} = \frac{3}{8} = \frac{x_0}{2}$$
Similarly, $x_2 = \frac{x_1}{2} = \frac{x_0}{2^2}$ and $x_3 = \frac{x_0}{2^3}$
Continuing this, we get, $x_{n-1} = \frac{x_0}{2^{n-1}}$ It may also be observed that, for each $f_n(x)$
$$x_0 = \frac{(1/2^n) + \left(1/2^{n+1}\right)}{2} = \frac{3}{2^{n+1}},$$
$x_n = \frac{x_0}{2^n} = \frac{3}{2^{2n+1}} \Rightarrow x_{n+1} = \frac{3}{2^{2n+2}}$ which is the zero of F. Therefore Newton's method converges to zero of F.

55. For small x we approximate e^x by $x+1$ (see exercise 44)
$$\frac{Le^{2\pi d/L} - e^{-2\pi d/L}}{e^{2\pi d/L} + e^{-2\pi d/L}}$$
$$\approx \frac{L\left[\left(1+\frac{2\pi d}{L}\right) - \left(1 - \frac{2\pi d}{L}\right)\right]}{\left(1+\frac{2\pi d}{L}\right) + \left(1-\frac{2\pi d}{L}\right)}$$
$$\approx \frac{L\left(\frac{4\pi d}{L}\right)}{2} = 2\pi d$$
$$f(d) \approx \frac{4.9}{\pi}\cdot 2\pi d = 9.8d$$

56. If $f(x) = \dfrac{8\pi hcx^{-5}}{e^{hc/(kTx)} - 1}$, then using the linear approximation we see that
$$f(x) \approx \frac{8\pi hcx^{-5}}{(1+\frac{hc}{kTx}) - 1} = 8\pi kTx^{-4}$$
as desired.

57. $W(x) = \dfrac{PR^2}{(R+x)^2}$, $x_0 = 0$
$$W'(x) = \frac{-2PR^2}{(R+x)^3}$$
$$L(x) = W(x_0) + W'(x_0)(x - x_0)$$
$$= \frac{PR^2}{(R+0)^2} + \left(\frac{-2PR^2}{(R+0)^3}\right)(x-0)$$
$$= P - \frac{2Px}{R}$$
$$L(x) = 120 - .01(120) = P - \frac{2Px}{R}$$
$$= 120 - \frac{2\cdot 120x}{R}$$
$$.01 = \frac{2x}{R}$$
$$x = .005R = .005(20{,}900{,}000)$$
$$= 104{,}500 \text{ ft}$$

58. If $m = m_0(1 - v^2/c^2)^{1/2}$, then $m' = (m_0/2)(1 - v^2/c^2)^{-1/2}(-2v/c^2)$, and $m' = 0$ when $v = 0$. The linear approximation is the constant function $m = m_0$ for small values v.

59. The only positive solution is 0.6407.

60. The smallest positive solution of the first equation is 0.132782, and for the second equation the smallest positive solution is 1, so the species modeled by the second equation is certain to go extinct. This is consistent with the models, since the expected number of offspring for the population modeled by the first equation is 2.2, while for the second equation it is only 1.3.

61. The linear approximation for the inverse tangent function at $x = 0$ is
$f(x) \approx f(0) + f'(0)(x-0)$
$\tan^{-1}(x) \approx \tan^{-1}(0) + \frac{1}{1+0^2}(x-0)$
$\tan^{-1}(x) \approx x$
Using this approximation,
$$\phi = \tan^{-1}\left(\frac{3[1-d/D] - w/2}{D-d}\right)$$
$$\phi \approx \frac{3[1-d/D] - w/2}{D-d}$$
If $d = 0$, then $\phi \approx \frac{3-w/2}{D}$. Thus, if w or D increase, then ϕ decreases.

62. $d'(\theta) = D(w/6\sin\theta)$
$d(0) = D(1 - w/6)$ so
$$d(\theta) \approx d(0) + d'(0)(\theta - 0)$$
$$= D(1-w/6) + 0(\theta) = D(1-w/6),$$
as desired.

3.2 Indeterminate Forms and L'Hôpital's Rule

1. $$\lim_{x\to -2}\frac{x+2}{x^2-4}$$
$$= \lim_{x\to -2}\frac{x+2}{(x+2)(x-2)}$$
$$= \lim_{x\to -2}\frac{1}{x-2} = -\frac{1}{4}$$

2. $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}$
$\displaystyle=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-1)}$
$\displaystyle=\lim_{x\to 2}\frac{x+2}{x-1}=4$

3. $\displaystyle\lim_{x\to\infty}\frac{3x^2+2}{x^2-4}$
$\displaystyle=\lim_{x\to\infty}\frac{3+\frac{2}{x^2}}{1-\frac{4}{x^2}}$
$\displaystyle=\frac{3}{1}=3$

4. $\displaystyle\lim_{x\to-\infty}\frac{x+1}{x^2+4x+3}$ is type $\dfrac{\infty}{\infty}$;

we apply L'Hôpital's Rule to get
$\displaystyle\lim_{x\to-\infty}\frac{1}{2x+4}=0.$

5. $\displaystyle\lim_{t\to 0}\frac{e^{2t}-1}{t}$ is type $\dfrac{0}{0}$;
we apply L'Hôpital's Rule to get
$\displaystyle\lim_{t\to 0}\frac{\frac{d}{dt}\left(e^{2t}-1\right)}{\frac{d}{dt}t}$
$\displaystyle\lim_{t\to 0}\frac{2e^{2t}}{1}=\frac{2}{1}=2$

6. $\displaystyle\lim_{t\to 0}\frac{\sin t}{e^{3t}-1}$ is type $\dfrac{0}{0}$;
we apply L'Hôpital's Rule to get
$\displaystyle\lim_{t\to 0}\frac{\frac{d}{dt}(\sin t)}{\frac{d}{dt}(e^{3t}-1)}=\lim_{t\to 0}\frac{\cos t}{3e^{3t}}=\frac{1}{3}$

7. $\displaystyle\lim_{t\to 0}\frac{\tan^{-1}t}{\sin t}$ is type $\dfrac{0}{0}$;
we apply L'Hôpital's Rule to get
$\displaystyle\lim_{t\to 0}\frac{\frac{d}{dt}\left(\tan^{-1}t\right)}{\frac{d}{dt}(\sin t)}=\lim_{t\to 0}\frac{1/(1+t^2)}{\cos t}=1$

8. $\displaystyle\lim_{t\to 0}\frac{\sin t}{\sin^{-1}t}$ is type $\dfrac{0}{0}$;

we apply L'Hôpital's Rule to get
$\displaystyle\lim_{t\to 0}\frac{\frac{d}{dt}(\sin t)}{\frac{d}{dt}\left(\sin^{-1}t\right)}=\lim_{t\to 0}\frac{\cos t}{1/(\sqrt{1-t^2})}=1$

9. $\displaystyle\lim_{x\to\pi}\frac{\sin 2x}{\sin x}$ is type $\dfrac{0}{0}$;

we apply L'Hôpital's Rule to get

$\displaystyle\lim_{x\to\pi}\frac{2\cos 2x}{\cos x}=\frac{2(1)}{-1}=-2.$

10. $\displaystyle\lim_{x\to-1}\frac{\cos^{-1}x}{x^2-1}$ is undefined (numerator goes to π, denominator goes to 0).

11. $\displaystyle\lim_{x\to 0}\frac{\sin x-x}{x^3}$ is type $\dfrac{0}{0}$;
we apply L'Hôpital's Rule thrice to get
$\displaystyle=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=\lim_{x\to 0}\frac{-\sin x}{6x}$
$\displaystyle=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}.$

12. $\displaystyle\lim_{x\to 0}\frac{\tan x-x}{x^3}$ is type $\dfrac{0}{0}$;
we apply L'Hôpital's Rule to get
$\displaystyle\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}.$

Apply L'Hôpital's Rule twice more to get
$\displaystyle\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}$
$\displaystyle=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x}{6}=\frac{1}{3}.$

13. $\displaystyle\lim_{t\to 1}\frac{\sqrt{t}-1}{t-1}=\lim_{t\to 1}\frac{\sqrt{t}-1}{t-1}\cdot\frac{\sqrt{t}+1}{\sqrt{t}+1}$
$\displaystyle=\lim_{t\to 1}\frac{(t-1)}{(t-1)\sqrt{t}+1}$
$\displaystyle=\lim_{t\to 1}\frac{1}{\sqrt{t}+1}=\frac{1}{2}$

14. $\displaystyle\lim_{t\to 1}\frac{\ln t}{t-1}$ is type $\dfrac{0}{0}$;

we apply L'Hôpital's Rule to get
$\displaystyle\lim_{t\to 1}\frac{\frac{d}{dt}(\ln t)}{\frac{d}{dt}(t-1)}=\lim_{t\to 1}\frac{\frac{1}{t}}{1}=1$

15. $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^x}$ is type $\dfrac{\infty}{\infty}$;
we apply L'Hôpital's Rule thrice to get

$\displaystyle\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}$
$\displaystyle=\lim_{x\to\infty}\frac{6}{e^x}=0.$

16. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^4}$ is type $\dfrac{\infty}{\infty}$;

we apply L'Hôpital's Rule four times to get
$\displaystyle\lim_{x\to\infty}\frac{e^x}{4x^3}=\lim_{x\to\infty}\frac{e^x}{12x^2}$
$\displaystyle=\lim_{x\to\infty}\frac{e^x}{24x}=\lim_{x\to\infty}\frac{e^x}{24}=\infty.$

17. $\lim_{x\to 0}\dfrac{x\cos x-\sin x}{x\sin^2 x}$ is type $\dfrac{\infty}{\infty}$;
we apply L'Hôpital's Rule twice to get
$\lim_{x\to 0}\dfrac{\cos x-x\sin x-\cos x}{\sin^2 x+2x\sin x\cos x}$
$\displaystyle=\lim_{x\to 0}\frac{-x\sin x}{\sin x(\sin x+2x\cos x)}$

$$= \lim_{x\to 0} \frac{-x}{\sin x + 2x\cos x}$$
$$= \lim_{x\to 0} \frac{-1}{\cos x + 2\cos x - 2x\sin x}$$
$$= -\frac{1}{3}.$$

18. Rewrite as one fraction, we have
$$\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{x\cos x - \sin x}{x\sin x}\right)$$
which is of type $\frac{0}{0}$
we apply L'Hôpital's Rule to get
$$= \lim_{x\to 0}\left(\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}\right)$$
$$= \lim_{x\to 0}\left(\frac{\frac{d}{dx}(-x\sin x)}{\frac{d}{dx}(\sin x + x\cos x)}\right)$$
$$= \lim_{x\to 0}\left(\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}\right) = 0$$

19. Rewrite as one fraction, we have
$$\lim_{x\to 0}\left(\frac{x+1}{x} - \frac{2}{\sin 2x}\right)$$
$$= \lim_{x\to 0}\left(\frac{(x+1)\sin 2x - 2x}{x\sin 2x}\right) \text{ is type } \frac{0}{0};$$
we apply L'Hôpital's Rule four times to get
$$\lim_{x\to 0}\left(\frac{\frac{d}{dx}(x+1)\sin 2x - 2x}{\frac{d}{dx}(x\sin 2x)}\right)$$
$$= \lim_{x\to 0}\left(\frac{\sin 2x + 2(x+1)\cos 2x - 2}{\sin 2x + 2x\cos 2x}\right)$$
$$= \lim_{x\to 0}\left(\frac{\frac{d}{dx}(\sin 2x + 2(x+1)\cos 2x - 2)}{\frac{d}{dx}(\sin 2x + 2x\cos 2x)}\right)$$
$$= \lim_{x\to 0}\left(\frac{2\cos 2x + 2\cos 2x - 4(x+1)\sin 2x}{2\cos 2x + 2\cos 2x - 4x\sin 2x}\right)$$
$$= \frac{4}{4} = 1$$

20. $\lim_{x\to\frac{\pi}{2}}\left(\tan x + \frac{1}{x-\frac{\pi}{2}}\right)$
In this case the limit has the form $(\infty - \infty)$.
Rewrite $\tan x$ as $\frac{\sin x}{\cos x}$ and then as one fraction, we get
$$\lim_{x\to\frac{\pi}{2}}\left(\tan x + \frac{1}{x-\frac{\pi}{2}}\right)$$
$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{\sin x}{\cos x} + \frac{1}{x-\frac{\pi}{2}}\right)$$
$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{\left(x-\frac{\pi}{2}\right)\sin x + \cos x}{\left(x-\frac{\pi}{2}\right)\cos x}\right) \text{ is type } \frac{0}{0}$$
we apply L'Hôpital's Rule to get
$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{\sin x + \left(x-\frac{\pi}{2}\right)\cos x - \sin x}{\cos x - \left(x-\frac{\pi}{2}\right)\sin x}\right)$$
$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{\left(x-\frac{\pi}{2}\right)\cos x}{\cos x - \left(x-\frac{\pi}{2}\right)\sin x}\right) = 0$$

21. $\lim_{x\to\infty}\frac{\ln x}{x^2}$ is type $\frac{\infty}{\infty}$
we apply L'Hôpital's Rule to get
$$\lim_{x\to\infty}\frac{1/x}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0.$$

22. $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$ is type $\frac{\infty}{\infty}$;
we apply L'Hôpital's Rule to get
$$\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$$

23. $\lim_{t\to\infty}\frac{t}{e^t}$ is type $\frac{\infty}{\infty}$
we apply L'Hôpital's Rule to get
$$\lim_{t\to\infty}\frac{\frac{d}{dt}(t)}{\frac{d}{dt}(e^t)} = \lim_{t\to\infty}\frac{1}{e^t} = 0.$$

24. $\lim_{t\to\infty}\frac{\sin\frac{1}{t}}{\frac{1}{t}}$ is type $\frac{0}{0}$
we apply L'Hôpital's Rule to get
$$= \lim_{t\to\infty}\frac{-\frac{1}{t^2}\cos\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to\infty}\cos\frac{1}{t} = 1.$$

25. $\lim_{t\to 1}\frac{\ln(\ln t)}{\ln t}$
As t approaches ln from below, $\ln t$ is a small negative number. Hence $\ln(\ln t)$ is undefined, so the limit is undefined.

26. $\lim_{t\to 0}\left(\frac{\sin(\sin t)}{\sin t}\right)$ is type $\frac{0}{0}$
we apply L'Hôpital's Rule to get
$$\lim_{t\to 0}\left(\frac{\cos(\sin t)\cos t}{\cos t}\right) = 1.$$

27. $\lim_{x\to 0}\left(\frac{\sin(\sinh x)}{\sinh(\sin x)}\right)$ is type $\frac{0}{0}$
we apply L'Hôpital's Rule to get
$$\lim_{x\to 0}\left(\frac{\cos(\sinh x)\cosh x}{\cosh(\sin x)\cos x}\right) = 1$$

28. $\lim_{x\to 0}\left(\frac{\sin x - \sinh x}{\cos x - \cosh x}\right)$
$$= \lim_{x\to 0}\left(\frac{2\sin x - e^x + e^{-x}}{2\cos x - e^x - e^{-x}}\right)$$
$$= \lim_{x\to 0}\left(\frac{2e^x\sin x - e^{2x} + 1}{2e^x\cos x - e^{2x} - 1}\right) \text{ is type } \frac{0}{0}$$
we apply L'Hôpital's Rule twice to get
$$\lim_{x\to 0}\left(\frac{2e^x\cos x + 2e^x\sin x - 2e^{2x}}{-2e^x\sin x + 2e^x\cos x - 2e^{2x}}\right)$$
$$= \lim_{x\to 0}\left(\frac{\cos x + \sin x - 1}{\cos x - \sin x - 1}\right) \text{ is type } \frac{0}{0}$$
$$= \lim_{x\to 0}\left(\frac{-\sin x + \cos x}{-\sin x - \cos x}\right) = -1$$

29. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\cot x}$ is type $\dfrac{\infty}{\infty}$ we apply L'Hôpital's Rule to get

$$\lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \left(-\sin x \cdot \frac{\sin x}{x}\right) = (0)(1) = 0.$$

30. $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x}}{\ln x} = 0$ (numerator goes to 0 and denominator goes to $-\infty$).

31. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$

$$= \lim_{x\to\infty} \left(\left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$$

$$= \lim_{x\to\infty} \left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right)$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$$

32. $\lim\limits_{x\to\infty} \ln x - x = \lim\limits_{x\to\infty} \dfrac{\frac{\ln x}{x} - 1}{\frac{1}{x}} = -\infty$ since the numerator goes to -1 and the denominator goes to 0^+. (Recall Example 2.8 which shows $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$.)

33. Let $y = \left(1 + \dfrac{1}{x}\right)^x$

$\Rightarrow \ln y = x \ln\left(1 + \dfrac{1}{x}\right)$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

Hence $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln y} = e$.

34. Notice that the limit in question has the indeterminate form 1^∞. Also, note that as x gets large, $\left|\dfrac{x+1}{x-2}\right| = \dfrac{x+1}{x-2}$.

Define $y = \left(\dfrac{x+1}{x-2}\right)^{\sqrt{x^2-4}}$. Then

$\ln y = \sqrt{x^2-4} \ln\left(\dfrac{x+1}{x-2}\right)$ and

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \left(\sqrt{x^2-4} \ln\left(\frac{x+1}{x-2}\right)\right)$$

$$= \lim_{x\to\infty} \left(\frac{\ln\left(\frac{x+1}{x-2}\right)}{\frac{1}{\sqrt{x^2-4}}}\right)$$

This last limit has indeterminate form $\dfrac{0}{0}$, so we can apply L'Hôpital's Rule and simplify to find that the above is equal to

$\lim\limits_{x\to\infty} \dfrac{-3(x^2-4)^{3/2}}{-x^3+x^2+2x}$ and this is equal to 3. So $\lim\limits_{x\to\infty} \ln y = 3$.

Thus $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln y} = e^3 \approx 20.086$.

35. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{\sqrt{x}} - \dfrac{\sqrt{x}}{\sqrt{x+1}}\right)$

$$= \lim_{x\to 0^+} \left(\frac{\sqrt{x+1} - (\sqrt{x})^2}{\sqrt{x}\sqrt{x+1}}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{\sqrt{x+1} - x}{\sqrt{x}\sqrt{x+1}}\right)$$

$= \infty$.

36. $\lim\limits_{x\to 1} \dfrac{\sqrt{5-x}-2}{\sqrt{10-x}-3}$ is type $\dfrac{0}{0}$ we apply L'Hôpital's Rule to get

$$\lim_{x\to 1} \frac{\frac{1}{2}(5-x)^{-1/2}(-1)}{\frac{1}{2}(10-x)^{-1/2}(-1)}$$

$$= \lim_{x\to 1} \frac{\sqrt{10-x}}{\sqrt{5-x}} = \frac{3}{2}.$$

37. Let $y = (1/x)^x$. Then $\ln y = x\ln(1/x)$. Then $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} x\ln(1/x) = 0$, by Exercise 27. Thus $\lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} e^{\ln y} = 1$.

38. Let $y = \lim\limits_{x\to 0^+} (\cos x)^{1/x}$. Then

$$\ln y = \lim_{x\to 0^+} \frac{1}{x} \ln\cos x$$

$$= \lim_{x\to 0^+} \frac{\ln(\cos x)}{x} \text{ is type } \frac{0}{0}$$

so apply L'Hôpital's Rule to get

$$\lim_{x\to 0^+} \frac{-\tan x}{1} = 0.$$

Therefore the limit is $y = e^0 = 1$.

39. $\lim\limits_{t\to\infty} \left(\dfrac{t-3}{t+2}\right)^t = \lim\limits_{t\to\infty} \dfrac{(t-3)^t}{(t+2)^t}$

$$= \lim_{t\to\infty} \frac{\left(1-\frac{3}{t}\right)^t}{\left(1+\frac{2}{t}\right)^t} = \frac{\lim\limits_{t\to\infty} \left(1-\frac{3}{t}\right)^t}{\lim\limits_{t\to\infty} \left(1+\frac{2}{t}\right)^t}$$

$$= \frac{\lim\limits_{t\to\infty} \left(1+\frac{-3}{t}\right)^t}{\lim\limits_{t\to\infty} \left(1+\frac{2}{t}\right)^t} = \frac{e^{-3}}{e^2} = e^{-5}$$

40. $\lim\limits_{t\to\infty} \left(\dfrac{t-3}{2t+1}\right)^t = \lim\limits_{t\to\infty} \left(\dfrac{1-\frac{3}{t}}{2+\frac{1}{t}}\right)^t$

$$= \lim_{t\to\infty} \frac{\left(1-\frac{3}{t}\right)^t}{2^t\left(1+\frac{1/2}{t}\right)^t} = \lim_{t\to\infty} \frac{e^{-3}}{2^t e^{1/2}} = 0$$

41. L'Hôpital's rule does not apply. As $x \to 0$, the numerator gets close to 1 and the denominator is small and positive. Hence the limit is ∞.

42. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x^2}$ is type $\dfrac{0}{0}$, but $\lim\limits_{x\to 0} \dfrac{e^x}{2x}$ is not, so L'Hôpital's Rule does not apply to this limit.

43. L'Hôpital's rule does not apply. As $x \to 0$, the numerator is small and positive while the denominator goes to $-\infty$. Hence the limit is 0. Also $\lim\limits_{x\to 0} \dfrac{2x}{2/x}$, which equals $\lim\limits_{x\to 0} x^2$, is not of the form $\dfrac{0}{0}$ so L'Hôpital's rule doesn't apply here either.

44. $\lim\limits_{x\to 0} \dfrac{\sin x}{x^2}$ is type $\dfrac{0}{0}$, but $\lim\limits_{x\to 0} \dfrac{\cos x}{2x}$ is not, so L'Hôpital's rule does not apply. This limit is undefined because the numerator goes to 1 and the denominator goes to 0.

45. $\lim\limits_{x\to 0^+} \dfrac{\csc x}{\sqrt{x}}$

In this case limit has the form $\dfrac{\infty}{0}$, L'Hôspital's Rule should not be used.

46. $\lim\limits_{x\to 0^+} \dfrac{x^{-3/2}}{\ln x}$ is type $\dfrac{\infty}{-\infty}$. In this case L'Hôspital's Rule should be used.

47. $\lim\limits_{x\to\infty} \dfrac{x^2 - 3x + 1}{\tan^{-1} x} = \infty$. In this case limit has the form ∞. So L'Hôspital's Rule should not be used.

48. $\lim\limits_{x\to\infty} \dfrac{\ln\left(x^2\right)}{e^{x/3}}$ is type $\dfrac{\infty}{\infty}$. So L'Hôspital's Rule should be used.

49. (a) Starting with $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 2x}$, we cannot "cancel sin" to get $\lim\limits_{x\to 0} \dfrac{3x}{2x}$. We can cancel the x's in the last limit to get the final answser of $3/2$. The first step is likely to give a correct answer because the linear approximation of $\sin 3x$ is $3x$, and the linear approximation of $\sin 2x$ is $2x$. The linear approximations are better the closer x is to zero, so the limits are likely to be the same.

(b) $\lim\limits_{x\to 0} \dfrac{\sin nx}{\sin mx}$ is type $\frac{0}{0}$; we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{n\cos nx}{m\cos mx} = \frac{n}{m}.$$

50. (a) $$\lim_{x\to 0} \frac{\sin x^2}{x^2} = \lim_{x\to 0} \frac{2x\cos x^2}{2x} = \lim_{x\to 0} \cos x^2 = 1,$$

which is the same as $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

(b) $$\lim_{x\to 0} \frac{1-\cos x^2}{x^4} = \lim_{x\to 0} \frac{2x\sin x^2}{4x^3} = \lim_{x\to 0} \frac{\sin x^2}{2x^2} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x^2}{x^2} = \frac{1}{2} \text{ (by part (a))},$$

while

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}(1) = \frac{1}{2}$$

so both of these limits are the same.

(c) Based on the patterns found in exercise 45, we should guess

$$\lim_{x\to 0} \frac{\sin x^3}{x^3} = 1 \text{ and } \lim_{x\to 0} \frac{1-\cos x^3}{x^6} = \frac{1}{2}.$$

51. (a) $\dfrac{(x+1)(2+\sin x)}{x(2+\cos x)}$

(b) $\dfrac{x}{e^x}$

(c) $\dfrac{3x+1}{x-7}$

(d) $\dfrac{3-8x}{1+2x}$

52. (a) $\lim\limits_{x\to\infty} x - \ln x = \infty$ (see exercise 32).

(b) $\lim\limits_{x\to\infty} \sqrt{x^2+1} - x = 0$ (see exercise 31).

(c) $$\lim_{x\to\infty} \sqrt{x^2+4x} - x = \lim_{x\to\infty} (\sqrt{x^2+4x} - x) = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty} \frac{4x\frac{1}{x}}{(\sqrt{x^2+4x}+x)\frac{1}{x}} = \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{4}{x}}+1} = 2,$$

where to get from the second to the third line, we have multiplied by $\dfrac{(\sqrt{x^2+4x}+x)}{(\sqrt{x^2+4x}+x)}$.

53. $\lim_{x\to\infty} e^x = \lim_{x\to\infty} x^n = \infty$

$\lim_{x\to\infty} \dfrac{e^x}{x^n} = \infty$. Since n applications of L'Hôpital's rule yields

$\lim_{x\to\infty} \dfrac{e^x}{n!} = \infty$.

Hence e^x dominates x^n.

54. $\lim_{x\to\infty} \ln x = \lim_{x\to\infty} x^p = \infty$.

$\lim_{x\to\infty} \dfrac{\ln x}{x^p}$ is of type $\dfrac{\infty}{\infty}$

we use L'Hôpital's Rule to get

$\lim_{x\to\infty} \dfrac{\frac{1}{x}}{px^{p-1}} = \lim_{x\to\infty} \dfrac{1}{px^p} = 0$ (since $p > 0$).

Therefore, x^p dominates $\ln x$.

55. $\lim_{t\to\infty} \left(e^{\frac{t}{2}} - t^3\right)$ Since $e^{\frac{t}{2}}$ dominates t^3. So $\lim_{t\to\infty} \left(e^{\frac{t}{2}} - t^3\right) = \infty$

56. $\lim_{x\to\infty} \left(\dfrac{\sqrt{x} - \ln x}{\sqrt{x}}\right)$ is type $\dfrac{\infty}{\infty}$.

we apply L'Hôpital's Rule to get

$\lim_{x\to\infty} \left(\dfrac{\frac{1}{2\sqrt{x}} - \frac{1}{x}}{\frac{1}{2\sqrt{x}}}\right) = \lim_{x\to\infty} \left(\dfrac{x - 2\sqrt{x}}{x}\right)$

$= \lim_{x\to\infty} \left(1 - \dfrac{2}{\sqrt{x}}\right) = 1$.

57. $\lim_{x\to\infty} \dfrac{\ln\left(x^3 + 2x + 1\right)}{\ln\left(x^2 + x + 2\right)}$

we apply L'Hôpital's Rule

$\lim_{x\to\infty} \left(\dfrac{\frac{d}{dx}\left(\ln\left(x^3 + 2x + 1\right)\right)}{\frac{d}{dx}\left(\ln\left(x^2 + x + 2\right)\right)}\right)$

$= \lim_{x\to\infty} \left(\dfrac{\frac{3x^2+2}{x^3+2x+1}}{\frac{2x+1}{x^2+x+2}}\right)$

$= \lim_{x\to\infty} \left(\dfrac{3x^4 + 3x^3 + 8x^2 + 2x + 4}{2x^4 + x^3 + 4x^2 + 4x + 1}\right) = \dfrac{3}{2}$

In general, for numerator and denominator the highest degee of polynomials p and q, such that $p(x) > 0$ and $q(x) > 0$ for $x > 0$, should be the $\lim_{x\to\infty} \frac{\ln(p(x))}{\ln(q(x))}$.

58. $\lim_{x\to\infty} \dfrac{\ln\left(e^{3x} + x\right)}{\ln\left(e^{2x} + 4\right)}$ is $\dfrac{\infty}{\infty}$;

we apply L'Hôpital's Rule

$\lim_{x\to\infty} \left(\dfrac{\frac{d}{dx}\left(\ln\left(e^{3x} + x\right)\right)}{\frac{d}{dx}\left(\ln\left(e^{2x} + 4\right)\right)}\right)$

$= \lim_{x\to\infty} \left(\dfrac{\frac{3e^{3x}+1}{e^{3x}+x}}{\frac{2e^{2x}}{e^{2x}+4}}\right)$

$= \lim_{x\to\infty} \left(\dfrac{3e^{5x} + 12e^{3x} + e^{2x} + 4}{2e^{5x} + 2xe^{2x}}\right) = \dfrac{3}{2}$

In general,when the degree of exponential term in the numerator and denominator are different, then the $\lim_{x\to\infty} \dfrac{\ln\left(e^{kx} + p(x)\right)}{\ln\left(e^{cx} + q(x)\right)}$ for polynomials p and q and positive numbers. k and c will be the fraction of degrees that is $\frac{k}{c}$.

59. If $x \to 0$, then $x^2 \to 0$, so if $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = L$, then $\lim_{x\to 0} \dfrac{f(x^2)}{g(x^2)} = L$ (but not conversely). If $a \neq 0$ or 1, then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ involves the behavior of the quotient near a, while $\lim_{x\to a} \dfrac{f(x^2)}{g(x^2)}$ involves the behavior of the quotient near the different point a^2.

60. Functions $f(x) = |x|$ and $g(x) = x$ work. $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$ does not exist as it approaches -1 from the left and it approaches 1 from the right, but $\lim_{x\to 0} \dfrac{f(x^2)}{g(x^2)} = 1$.

61. $\lim_{\omega\to 0} \dfrac{2.5(4\omega t - \sin 4\omega t)}{4\omega^2}$

$= \lim_{\omega\to 0} \dfrac{2.5(4t - 4t\cos 4\omega t)}{8\omega}$

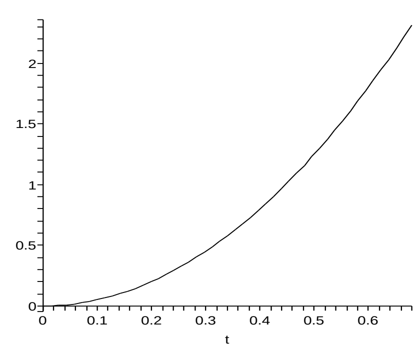
$= \lim_{\omega\to 0} \dfrac{2.5(16t^2 \sin 4\omega t)}{8} = 0$

62. $\lim_{\omega\to 0} \dfrac{2.5 - 2.5\sin(4\omega t + \dfrac{\pi}{2})}{4\omega^2}$ is type $\frac{0}{0}$;

we apply L'Hôpital's Rule to get

$\lim_{\omega\to 0} \dfrac{-10t\cos(4\omega t + \frac{\pi}{2})}{8\omega}$

$= \lim_{\omega\to 0} \dfrac{40t^2 \sin(4\omega t + \frac{\pi}{2})}{8} = 5t^2$.



63. The area of triangular region 1 is (1/2)(base)(height)

$= (1/2)(1-\cos\theta)(\sin\theta)$.
Let P be the center of the circle. The area of region 2 equals the area of sector APC minus the area of triangle APB. The area of the sector is $\theta/2$, while the area of triangle APB is
$(1/2)(\text{base})(\text{height})$
$= (1/2)(\cos\theta)(\sin\theta)$.
Hence the area of region 1 divided by the area of region 2 is

$$\frac{(1/2)(1-\cos\theta)(\sin\theta)}{\theta/2-(1/2)(\cos\theta)(\sin\theta)}$$
$$= \frac{(1-\cos\theta)(\sin\theta)}{\theta-\cos\theta\sin\theta}$$
$$= \frac{\sin\theta-\cos\theta\sin\theta}{\theta-\cos\theta\sin\theta}$$
$$= \frac{\sin\theta-(1/2)\sin 2\theta}{\theta-(1/2)\sin 2\theta}$$

Then $\lim_{\theta\to 0} \dfrac{\sin\theta-(1/2)\sin 2\theta}{\theta-(1/2)\sin 2\theta}$

$$= \lim_{\theta\to 0}\frac{\cos\theta-\cos 2\theta}{1-\cos 2\theta}$$
$$= \lim_{\theta\to 0}\frac{-\sin\theta+2\sin 2\theta}{2\sin 2\theta}$$
$$= \lim_{\theta\to 0}\frac{-\cos\theta+4\cos 2\theta}{4\cos 2\theta}$$
$$= \frac{-1+4(1)}{4(1)} = \frac{3}{4}$$

64. $\lim_{x\to 0^+}\dfrac{160x^{-0.4}+90}{8x^{-0.4}+10}$
$= \lim_{x\to 0^+}\dfrac{160+90x^{0.4}}{8+10x^{0.4}} = \dfrac{160}{8} = 20$. If there is no light, the pupils will expand to this size. This is the largest the pupils can get. $\lim_{x\to\infty}\dfrac{160x^{-0.4}+90}{8x^{-0.4}+10} = \dfrac{90}{10} = 9$. As the amount of light grows, the pupils shrink, and the size approaches 6mm in the limit. This is the smallest possible size of the pupils.

65. (a) $V = \sqrt{40mg}\tanh\left(\sqrt{\frac{g}{40m}}\,t\right)$, therefore $\lim_{t\to\infty} V$

$$= \lim_{t\to\infty}\sqrt{40mg}\left(\frac{e^{\sqrt{\frac{g}{40m}}\,t}-e^{-\sqrt{\frac{g}{40m}}\,t}}{e^{\sqrt{\frac{g}{40m}}\,t}+e^{-\sqrt{\frac{g}{40m}}\,t}}\right)$$
$$= \sqrt{40mg}\lim_{t\to\infty}\left(\frac{1-e^{-2\sqrt{\frac{g}{40m}}\,t}}{1+e^{-2\sqrt{\frac{g}{40m}}\,t}}\right)$$
$$= \sqrt{40mg}$$

as $t\to\infty$; $2\sqrt{\dfrac{g}{40m}}t\to\infty$ and $e^{-2\sqrt{\frac{g}{40m}}t}\to 0$ This means, when the time increases indefinitely, its velocity reaches $\sqrt{40mg}$.

(b) $\lim_{m\to 0} V$

$$= \lim_{m\to 0}\sqrt{40mg}\left(\frac{e^{\sqrt{\frac{g}{40m}}\,t}-e^{-\sqrt{\frac{g}{40m}}\,t}}{e^{\sqrt{\frac{g}{40m}}\,t}+e^{-\sqrt{\frac{g}{40m}}\,t}}\right)$$
$$= \lim_{m\to 0}\sqrt{40mg}\left(\frac{1-e^{-2\sqrt{\frac{g}{40m}}\,t}}{1+e^{-2\sqrt{\frac{g}{40m}}\,t}}\right)$$
$$= \lim_{m\to 0}\sqrt{40mg}\lim_{m\to 0}\left(\frac{1-e^{-2\sqrt{\frac{g}{40m}}\,t}}{1+e^{-2\sqrt{\frac{g}{40m}}\,t}}\right)$$
$$= 0$$

as $m\to 0$; $2\sqrt{\dfrac{g}{40m}}\,t\to\infty$ and $e^{-2\sqrt{\frac{g}{40m}}t}\to 0$. This means, when the mass is negligible, its velocity is 0.

(c) $\lim_{m\to\infty} V$

$$= \lim_{m\to\infty}\sqrt{40mg}\left(\frac{e^{\sqrt{\frac{g}{40m}}\,t}-e^{-\sqrt{\frac{g}{40m}}\,t}}{e^{\sqrt{\frac{g}{40m}}\,t}+e^{-\sqrt{\frac{g}{40m}}\,t}}\right)$$
$$= \lim_{m\to\infty}\sqrt{40mg}\left(\frac{e^{2\sqrt{\frac{g}{40m}}\,t}-1}{e^{2\sqrt{\frac{g}{40m}}\,t}+1}\right)$$

as $m\to\infty$; $2\sqrt{\dfrac{g}{40m}}\,t\to 0$ and $e^{2\sqrt{\frac{g}{40m}}\,t}\to 1$

$$= \lim_{m\to\infty}\left(\frac{1}{e^{2\sqrt{\frac{g}{40m}}\,t}+1}\right)\lim_{m\to\infty}\frac{\left(e^{2\sqrt{\frac{g}{40m}}\,t}-1\right)}{(1/\sqrt{40mg})}$$
$$= (1/2)\lim_{\left(2\sqrt{\frac{g}{40m}}\,t\right)\to 0}\left(\frac{e^{2\sqrt{\frac{g}{40m}}\,t}-1}{2\sqrt{\frac{g}{40m}}\,t}\right)\left(2\sqrt{g}\,t\right)$$
$$= \sqrt{g}\,t$$

This means, when the increases indefinitely, its velocity reaches $\sqrt{g}t$.

66. $\lim_{c\to\infty} S = \lim_{c\to\infty}\left\{\frac{8\pi}{3}c^2\left[\left(\frac{d^2}{16c^2}+1\right)^{3/2}-1\right]\right\}$

$$= \frac{8\pi}{3}\lim_{c\to\infty}\left(\frac{c^2\left[\left(d^2+16c^2\right)^{3/2}-64c^3\right]}{64c^3}\right)$$
$$= \frac{\pi}{24}\lim_{c\to\infty}\left(\frac{\left[\left(d^2+16c^2\right)^{3/2}-64c^3\right]}{c}\right)$$ is type $\frac{\infty}{\infty}$;

we apply L'Hôpital's Rule to get

$$\frac{\pi}{24}\lim_{c\to\infty}\left[\frac{3}{2}\left(d^2+16c^2\right)^{1/2}(32c)-192c^2\right]$$
$$= 2\pi\lim_{c\to\infty} c\left[\left(d^2+16c^2\right)^{1/2}-4c^2\right]$$

which on rationalising gives

$$2\pi\lim_{c\to\infty}\frac{c\left[\left(d^2+16c^2\right)-16c^2\right]}{\left[\left(d^2+16c^2\right)^{1/2}+4c^2\right]} = \frac{\pi d^2}{4}$$

3.3 Maximum and Minimum Values

1. (a) $f(x) = \frac{1}{x^2-1}$ on $(0, 1) \cup (1, \infty)$
$f'(x) = \frac{-2x}{(x^2-1)^2}$
$x = 0$ is critical point.
$f(0) = -1$ is absolute maximum value but 0 is not included. Hence f has no absolute extrema on interval $(0, 1) \cup (1, \infty)$.

(b) $f(x) = \frac{1}{x^2-1}$ on (-1, 1)
$f'(x) = \frac{-2x}{(x^2-1)^2}$
$x = 0$ is the only critical point.
$f(0) = -1$ is absolute maximum value of $f(x)$. Hence f has no absolute minimum on interval $(-1, 1)$

(c) No absolute extrema. (They would be at the endpoints which are not included in the interval.)

(d) $f(x) = \frac{1}{x^2-1}$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$
$f'(x) = \frac{-2x}{(x^2-1)^2}$
$x = 0$ is critical point.
f has an absolute maximum value of $f(0) = -1$. f assumes its minimum at two points $x = \pm\frac{1}{2}$ and minimum value is $f\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = -\frac{4}{3}$.

2. (a) $f(x) = \frac{x^2}{(x-1)^2}$ on $(-\infty, 1) \cup (1, \infty)$
$f'(x) = \frac{2x(x-1)^2 - 2x^2(x-1)}{(x-1)^4} = 0$
$x = 0$ is critical point.
f has an absolute minimum value of $f(0) = 0$ at $x = 0$ and no absolute maximum occurs.

(b) $f(x) = \frac{x^2}{(x-1)^2}$ on $(-1, 1)$
$f'(x) = \frac{2x(x-1)^2 - 2x^2(x-1)}{(x-1)^4} = 0$
$x = 0$ is critical point.
f has an absolute minimum value $f(0) = 0$ at $x = 0$ and there is no absolute maximum.

(c) The function does not have a maximum or minimum. The minimum would be at $x = 0$ (not included in this interval) while the asymptote at $x = 1$ precludes an absolute maximum.

(d) $f(x) = \frac{x^2}{(x-1)^2}$ on $[-2, -1]$
$f'(x) = \frac{2x(x-1)^2 - 2x^2(x-1)}{(x-1)^4}$
$= \frac{-2x(x-1)}{(x-1)^4} < 0$ on $[-2, -1]$
$f(x)$ is decreasing function on $[-2, -1]$.
$f(x)$ is maximum at $x = -2$ and minimum at $x = -1$.

3. (a) $f(x) = x^2 + 5x - 1$
$f'(x) = 2x + 5$
$2x + 5 = 0$
$x = -5/2$ is a critical number.
This is a parabola opening upward, so we have a minimum at $x = -5/2$.

(b) $f(x) = -x^2 + 4x + 2$
$f'(x) = -2x + 4 = 0$ when $x = 2$.
This is a parabola opening downward, so we have a maximum at $x = 2$.

4. (a) $f(x) = x^3 - 3x + 1$
$f'(x) = 3x^2 - 3$
$= 3(x^2 - 1)$
$= 3(x+1)(x-1) = 0$
$x = \pm 1$ are critical numbers and $f(1) = -1$, $f(-1) = 3$.
This is a cubic with a positive leading coefficient so $x = -1$ is a local max, $x = 1$ is a local min.

(b) $f(x) = -x^3 + 6x^2 + 2$
$f'(x) = -3x^2 + 12x = -3x(x+4) = 0$ when $x = 0$ and $x = -4$.
$f(0) = 2$, $f(-4) = 162$.
This is a cubic with a negative leading coefficient so $x = 0$ is a local min and $x = -4$ is a local max.

5. (a) $f(x) = x^3 - 3x^2 + 6x$
$f'(x) = 3x^2 - 6x + 6$
$3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 0$
We can use the quadratic formula to find the roots, which are $x = 1 \pm \sqrt{-1}$. These are imaginary so there are no real critical points.

(b) $f(x) = -x^3 + 3x^2 - 3x$
$f'(x) = -3x^2 + 6x - 3$
$= 3\left(-x^2 + 2x - 1\right)$
$= -3\left(x^2 - 2x + 1\right)$
$= -3(x-1)^2$

$f'(x) = 3(x-1)^2 = 0$ when $x = 1$.
Since $f(x)$ is a cubic with only one critical number it is neither local min nor max.

6. (a) $f(x) = x^4 - 2x^2 + 1$

$$f'(x) = 4x^3 - 4x$$
$$= 4x\left(x^2 - 1\right)$$
$$= 4x(x-1)(x+1)$$

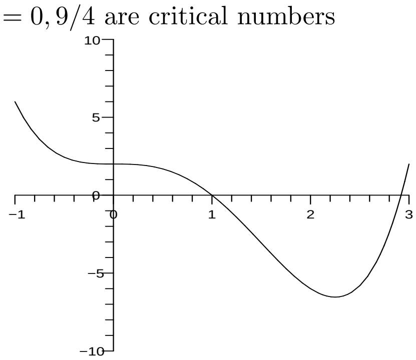
$f'(x) = 0$ when $x = 0, \pm 1$.
$x = 0, \pm 1$ are critical numbers. $x = 0$ is local maximum and $x = \pm 1$ are local minimum.

(b) $f(x) = x^4 - 3x^3 + 2$

$$f'(x) = 4x^3 - 9x^2$$
$$= x^2(4x - 9)$$

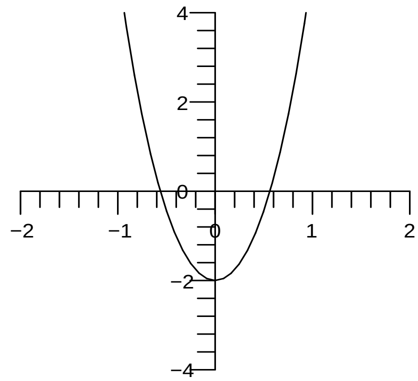
$f'(x) = 0$ when $x = 0, \frac{9}{4}$.
$x = 0, \frac{9}{4}$ are critical points. $x = \frac{9}{4}$ is local minimum and $x = 0$ is neither max nor min.

7. $f(x) = x^4 - 3x^3 + 2$
$f'(x) = 4x^3 - 9x^2$
$4x^3 - 9x^2 = x^2(4x - 9) = 0$
$x = 0, 9/4$ are critical numbers



$x = 9/4$ is a local min; $x = 0$ is neither a local max nor min.

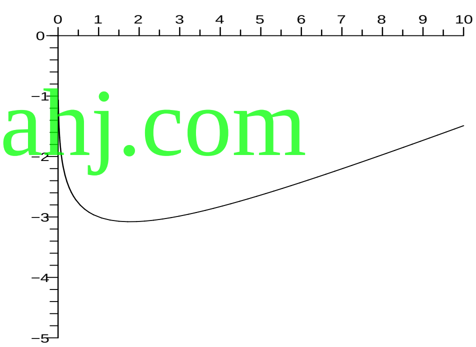
8. $f(x) = x^4 + 6x^2 - 2$
$f'(x) = 4x^3 + 12x = 0$ when $x = 0$ (minimum).



9. $f(x) = x^{3/4} - 4x^{1/4}$

$$f'(x) = \frac{3}{4x^{1/4}} - \frac{1}{x^{3/4}}$$

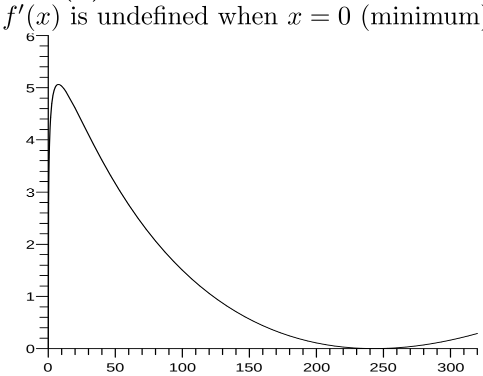
If $x \neq 0$, $f'(x) = 0$ when $3x^{3/4} = 4x^{1/4}$
$x = 0,\ 16/9$ are critical numbers.
$x = 16/9$ is a local min, $x = 0$ is a local maximum.



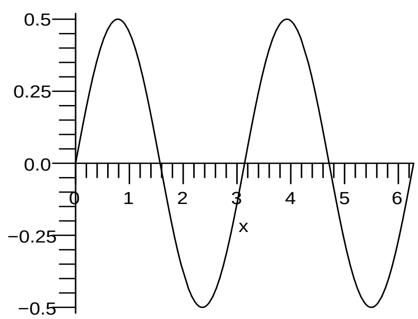
10. $f(x) = (x^{2/5} - 3x^{1/5})^2$

$$f'(x) = 2(x^{2/5} - 3x^{1/5})\left(\frac{2}{5x^{3/5}} - \frac{3}{5x^{4/5}}\right)$$

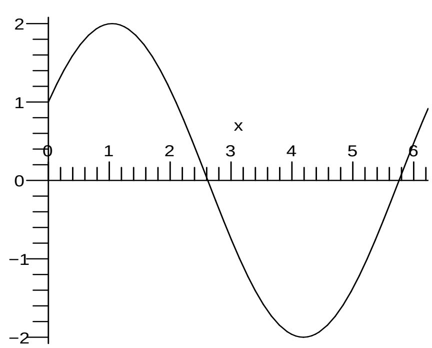
$f'(x) = 0$ when $x = 3^5$ (minimum) and $x = \left(\frac{3}{2}\right)^5$ (maximum).
$f'(x)$ is undefined when $x = 0$ (minimum).



11. $f(x) = \sin x \cos x$ on $[0, 2\pi]$
$f'(x) = \cos x \cos x + \sin x(-\sin x)$
$\quad = \cos^2 x - \sin^2 x$
$\cos^2 x - \sin^2 x = 0$
$\cos^2 x = \sin^2 x$
$\cos x = \pm \sin x$
$x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$
are critical numbers.
$x = \pi/4, 5\pi/4$ are local max, $x = 3\pi/4, 7\pi/4$ are local min.
Also $x = 0$ is local minimum and $x = 2\pi$ is local maximum.



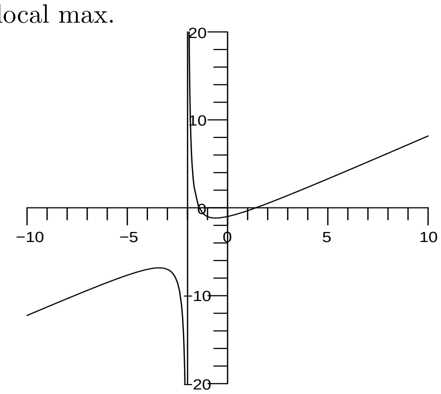
12. $f(x) = \sqrt{3}\sin x + \cos x$
$f'(x) = \sqrt{3}\cos x - \sin x = 0$ when $\tan(x) = \sqrt{3}$ or $x = \pi/3 + k\pi$ for any integer k (maxima for even k and minima for odd k).



13. $f(x) = \dfrac{x^2 - 2}{x + 2}$
Note that $x = -2$ is not in the domain of f.

$$f'(x) = \frac{(2x)(x+2) - (x^2 - 2)(1)}{(x+2)^2} = \frac{2x^2 + 4x - x^2 + 2}{(x+2)^2} = \frac{x^2 + 4x + 2}{(x+2)}$$

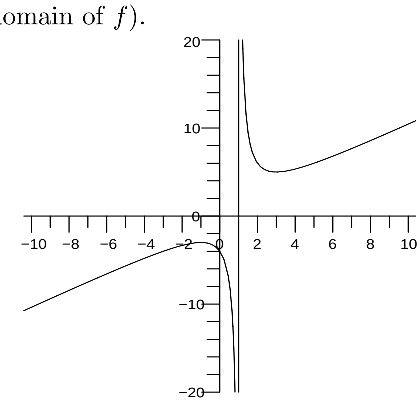
$f'(x) = 0$ when $x^2 + 4x + 2 = 0$, so the critical numbers are $x = -2 \pm \sqrt{2}$.
$x = -2 + \sqrt{2}$ is a local min; $x = -2 + \sqrt{2}$ is a local max.



14. $f(x) = \dfrac{x^2 - x + 4}{x - 1}$

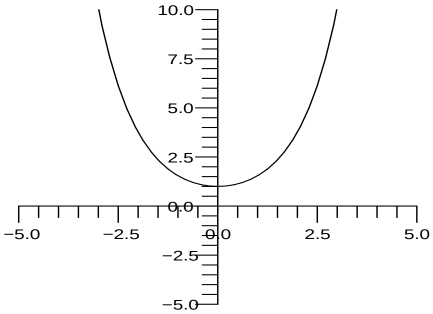
$$f'(x) = \frac{(x-1)(2x-1) - (x^2 - x + 4)}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2} = 0$$

when $x = -1$ (maximum) and $x = 3$ (minimum). $f'(x)$ is undefined when $x = 1$ (not in domain of f).

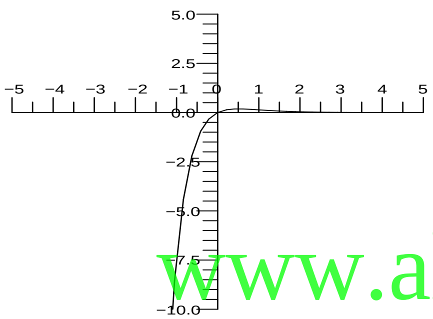


15. $f(x) = \dfrac{e^x + e^{-x}}{2}$

$f'(x) = \dfrac{e^x - e^{-x}}{2}$
$f'(x) = 0$ when $e^x = e^{-x}$, that is, $x = 0$.
$f'(x)$ is defined for all x, so $x = 0$ is a critical number. $x = 0$ is a local min.



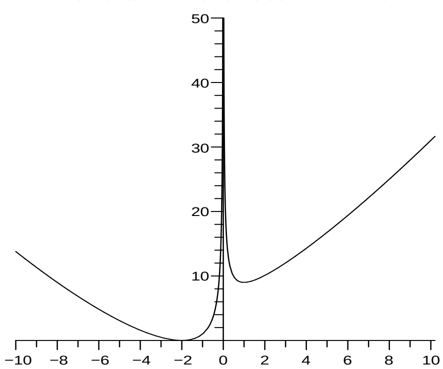
16. $f(x) = xe^{-2x}$
$f'(x) = e^{-2x} - 2xe^{-2x} = 0$ when $x = \frac{1}{2}$ (maximum).



17. $f(x) = x^{4/3} + 4x^{1/3} + 4x^{-2/3}$
f is not defined at $x = 0$.

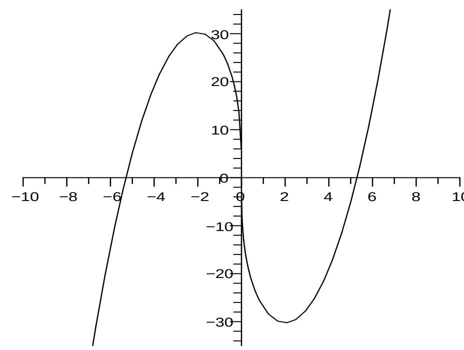
$$\begin{aligned} f'(x) &= \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} - \frac{8}{3}x^{-5/3} \\ &= \frac{4}{3}x^{-5/3}(x^2 + x - 2) \\ &= \frac{4}{3}x^{-5/3}(x-1)(x+2) \end{aligned}$$

$x = -2, 1$ are critical numbers.
$x = -2$ and $x = 1$ are local minima.



18. $f(x) = x^{7/3} - 28x^{1/3}$
$f'(x) = \dfrac{7}{3}x^{4/3} - \dfrac{28}{3}x^{-2/3} = 0$ when $x = -2$ (local maximum) and $x = 2$ (local minimum).

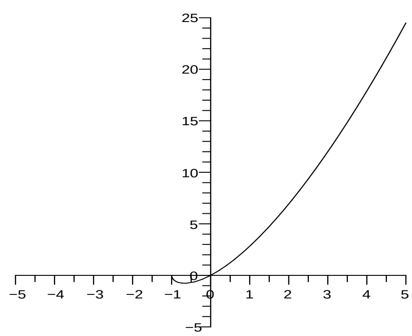
$f'(x)$ is undefined at $x = 0$ (neither)



19. $f(x) = 2x\sqrt{x+1} = 2x(x+1)^{1/2}$
Domain of f is all $x \geq -1$.

$$\begin{aligned} f'(x) &= 2(x+1)^{1/2} + 2x\left(\tfrac{1}{2}(x+1)^{-1/2}\right) \\ &= \frac{2(x+1) + x}{\sqrt{x+1}} \\ &= \frac{3x+2}{\sqrt{x+1}} \end{aligned}$$

$f'(x) = 0$ for $3x + 2 = 0$, $x = -2/3$.
$x = -2/3$ is critical numbers.
$f'(x)$ is undefined for $x = -1$.

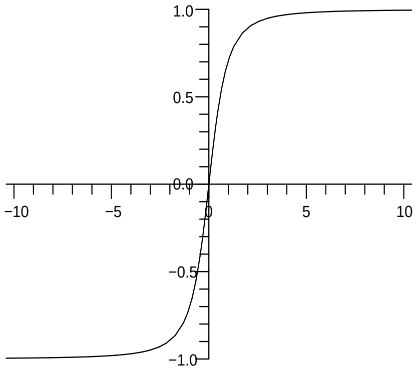


$x = -2/3$ is a local min. $x = -1$ is an endpoint and local maximum.

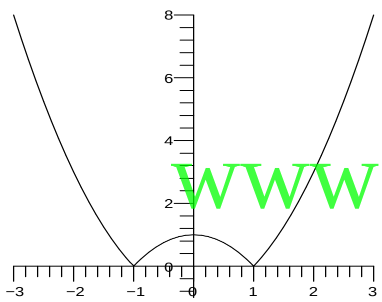
20. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$$\begin{aligned} f'(x) &= \frac{\sqrt{x^2+1} - \dfrac{x^2}{\sqrt{x^2+1}}}{x^2+1} \\ &= \frac{1}{(x^2+1)^{3/2}} \neq 0 \text{ for any } x, \text{ and } f(x) \end{aligned}$$

has no critical points.



21. Because of the absolute value sign, there may be critical numbers where the function $x^2 - 1$ changes sign; that is, at $x = \pm 1$. For $x > 1$ and for $x < -1$, $f(x) = x^2 - 1$ and $f'(x) = 2x$, so there are no critical numbers on these intervals. For $-1 < x < 1$, $f(x) = 1 - x^2$ and $f'(x) = -2x$, so 0 is a critical number.



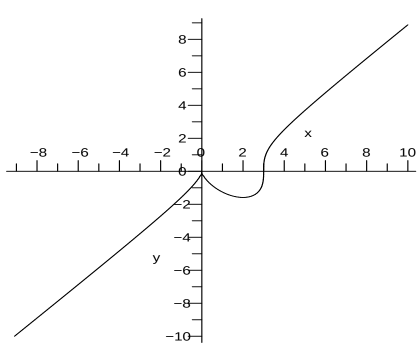
The graph confirms this analysis and shows there is a local max at $x = 0$ and local min at $x = \pm 1$.

22. $f(x) = \sqrt[3]{(x^3 - 3x^2)} = \left(x^3 - 3x^2\right)^{\frac{1}{3}}$

$f'(x) = \dfrac{1}{3} \cdot \dfrac{3x^2 - 6x}{(x^3 - 3x^2)^{\frac{2}{3}}} = \dfrac{1}{3} \cdot \dfrac{3x^2 - 6x}{(x^3 - 3x^2)^{\frac{2}{3}}} = 0$

when $x = 2$.

$x = 2$ is critical number. $x = 2$ is local minimum. $x = 0$ is local maximum.



23. First, let's find the critical numbers for $x < 0$. In this case,

$f(x) = x^2 + 2x - 1$

$f'(x) = 2x + 2 = 2(x + 1)$

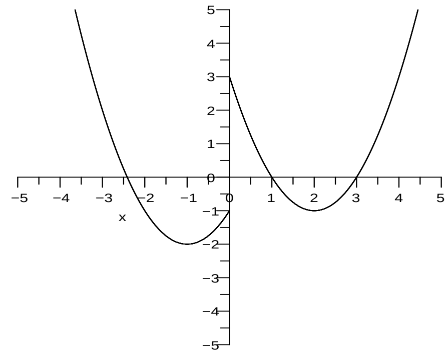
so the only critical number in this interval is $x = -1$ and it is a local minimum.

Now for $x > 0$,

$f(x) = x^2 - 4x + 3$

$f'(x) = 2x - 4 = 2(x - 2)$

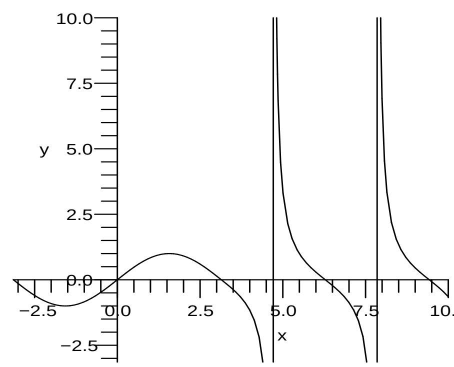
so the only critical number is $x = 2$ and it is a local minimum.



Finally, since f is not continuous and hence not differentiable at $x = 0$. Indeed, $x = 0$ is a local maximum.

24. $f'(x) = \cos x$ for $-\pi < x < \pi$, and $f'(x) = -\sec^2 x$ for $|x| \geq \pi$.

$f'(x) = 0$ for $x = -\pi/2$ (minimum) and $x = \pi/2$ (maximum).



$f'(x)$ is undefined for $x = (2k+1)\frac{\pi}{2}$ for integers $k \neq -1$ or 0 (not in domain of f).

25. $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$f'(x) = 0$ for $x = \pm 1$.

(a) On $[0, 2]$, 1 is the only critical number. We calculate:
$f(0) = 1$
$f(1) = -1$ is the abs min.
$f(2) = 3$ is the abs max.

(b) On the interval $[-3, 2]$, we have both 1 and -1 as critical numbers.
We calculate:
$f(-3) = -17$ is the abs min.
$f(-1) = 3$ is the abs max.
$f(1) = -1$
$f(2) = 3$ is also the abs max.

26. $f(x) = x^4 - 8x^2 + 2$
$f'(x) = 4x^3 - 16x = 0$ when $x = 0$ and $x = \pm 2$.

(a) On $[-3,\ 1]$:
$f(-3) = 11$, $f(-2) = -14$, $f(0) = 2$, and $f(1) = -5$.
The abs min on this interval is $f(-2) = -14$ and the abs max is $f(-3) = 11$.

(b) On $[-1,\ 3]$:
$f(-1) = -5$, $f(2) = -14$, and $f(3) = 11$.
The abs min on this interval is $f(2) = -14$ and the abs max is $f(3) = 11$.

27. $f(x) = x^{2/3}$
$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$
$f'(x) \neq 0$ for any x, but $f'(x)$ undefined for $x = 0$, so $x = 0$ is critical number.

(a) On $[-4,\ -2]$:
$0 \notin [-4, -2]$ so we only look at endpoints.
$f(-4) = \sqrt[3]{16} \approx 2.52$
$f(-2) = \sqrt[3]{4} \approx 1.59$
So $f(-4) = \sqrt[3]{16}$ is the abs max and $f(-2) = \sqrt[3]{4}$ is the abs min.

(b) On $[-1,\ 3]$, we have 0 as a critical number.
$f(-1) = 1$
$f(0) = 0$ is the abs min.
$f(3) = 3^{2/3}$ is the abs max.

28. $f(x) = \sin x + \cos x$
$f'(x) = \cos x - \sin x = 0$ when $x = \frac{\pi}{4} + k\pi$ for integers k.

(a) On $[0,\ 2\pi]$:
$f(0) = 1$, $f(\pi/4) = \sqrt{2}$, $f(5\pi/4) = -\sqrt{2}$, and $f(2\pi) = 1$.
The abs min on this interval is $f(5\pi/4) = -\sqrt{2}$ and the abs max is $f(\pi/4) = \sqrt{2}$.

(b) On $[\pi/2,\ \pi]$:
$f(\pi/2) = 1$, $f(\pi) = -1$.
The abs min on this interval is $f(\pi) = -1$ and the abs max is $f(\pi/2) = 1$.

29. $f(x) = e^{-x^2}$
$f'(x) = -2xe^{-x^2}$
Hence $x = 0$ is the only critical number.

(a) On $[0,\ 2]$:
$f(0) = 1$ is the abs max.
$f(2) = e^{-4}$ is the abs min.

(b) On $[-3,\ 2]$:
$f(-3) = e^{-9}$ is the abs min.
$f(0) = 1$ is the abs max.
$f(2) = e^{-4}$

30. $f(x) = x^2 e^{-4x}$
$f'(x) = 2xe^{-4x} - 4x^2 e^{-4x} = 0$ when $x = 0$ and $x = 1/2$.

(a) On $[-2,\ 0]$:
$f(-2) = 4e^8$, $f(0) = 0$.
The abs min is $f(0) = 0$ and the abs max is $f(-2) = 4e^8$.

(b) On $[0,\ 4]$:
$f(1/2) = e^{-2}/4$, $f(4) = 16e^{-16}$.
The abs min is $f(0) = 0$ and the abs max is $f(1/2) = e^{-2}/4$.

31. $f(x) = \dfrac{3x^2}{x-3}$
Note that $x = 3$ is not in the domain of f.

$$f'(x) = \frac{6x(x-3) - 3x^2(1)}{(x-3)^2} = \frac{6x^2 - 18x - 3x^2}{(x-3)^2} = \frac{3x^2 - 18x}{(x-3)^2} = \frac{3x(x-6)}{(x-3)^2}$$

The critical points are $x = 0$, $x = 6$.

(a) On $[-2, 2]$:
$f(-2) = -12/5$
$f(2) = -12$
$f(0) = 0$
Hence abs max is $f(0) = 0$ and abs min is $f(2) = -12$.

(b) On $[2, 8]$, the function is not continuous and in fact has no absolute max or min.

32. $f(x) = \tan^{-1}(x^2)$
$f'(x) = \dfrac{2x}{1+x^4} = 0$ when $x = 0$.

(a) On $[0, 1]$:
$f(0) = 0$ and $f(1) = \pi/4$.
The abs min is $f(0) = 0$ and the abs max is $f(1) = \pi/4$.

(b) On $[-3, 4]$:
$f(-3) \approx 1.46$, $f(0) = 0$, and $f(4) \approx 1.51$.
The abs min is $f(0) = 0$ and the abs max is $f(4) = \tan^{-1} 16$.

33. $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{\left(x^2+1\right)\cdot 1 - x\cdot(2x)}{\left(x^2+1\right)^2}$$
$$= \frac{\left(x^2+1\right)\cdot 1 - x\cdot(2x)}{\left(x^2+1\right)^2} = \frac{-x^2+1}{\left(x^2+1\right)^2} = 0$$

when $x = \pm 1$.
$x = \pm 1$ are critical numbers.

(a) On [0, 2]:
$f(0) = \dfrac{0}{0^2+1} = 0$ is the abs minimum.
$f(2) = \dfrac{2}{2^2+1} = \dfrac{2}{5}$
$f(1) = \dfrac{1}{2}$ is the abs maximum.

(b) On [−3, 3]:
$f(3) = -\dfrac{3}{10}$
$f(-1) = -\dfrac{1}{2}$ is the abs minimum.
$f(1) = \dfrac{1}{2}$ is the abs maximum.
$f(3) = \dfrac{3}{10}$

34. $f(x) = \dfrac{3x}{x^2+16}$

$$f'(x) = \frac{\left(x^2+16\right)\cdot 3 - 3x\cdot(2x)}{\left(x^2+16\right)^2}$$
$$= \frac{\left(x^2+16\right)\cdot 3 - 3x\cdot(2x)}{\left(x^2+16\right)^2} = 0$$
$$= \frac{-3x^2+48}{\left(x^2+16\right)^2} = 0 \text{ when } x = \pm 4.$$

$x = \pm 4$ are critical numbers.

(a) On [0, 2]:
$f(0) = \dfrac{0}{0^2+16} = 0$ is the abs minimum.
$f(2) = \dfrac{2}{2^2+1} = \dfrac{3}{10}$ is the abs maximum.

(b) on [0, 6]:
$f(0) = 0$ is abs minimum.
$f(4) = \dfrac{3}{8}$ is abs maximum.
$f(6) = \dfrac{9}{26}$

35. $f'(x) = 4x^3 - 6x + 2 = 0$ at about $x = 0.3660$, -1.3660 and at $x = 1$.

(a) $f(-1) = -3$, $f(1) = 1$.
The absolute min is $(-1, -3)$ and the absolute max is approximately $(0.3660, 1.3481)$.

(b) The absolute min is approximately $(-1.3660, -3.8481)$ and the absolute max is $(-3, 49)$.

36. $f'(x) = 6x^5 - 12x - 2 = 0$ at about -1.3673, -0.5860 and 1.4522.

(a) $f(-1) = 1$, $f(1) = -3$. $f(-0.5860) = 1.8587$.
The absolute min is $f(1) = -3$ and the absolute max is approximately $f(-0.5860) = 1.8587$.

(b) $f(-2) = 21$ and $f(2) = 13$. $f(-1.3673) = -.2165$ and $f(1.4522) = -5.8675$.
The absolute min is approximately $f(1.4522) = -5.8675$ and the absolute max is $f(-2) = 21$.

37. $f'(x) = \sin x + x\cos x = 0$ at $x = 0$ and about 2.0288 and 4.9132.

(a) The absolute min is $(0, 3)$ and the absolute max is $(\pm\pi/2, 3 + \pi/2)$.

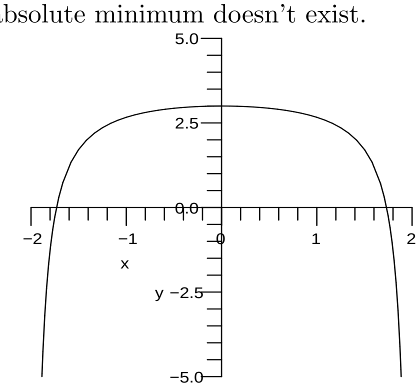
(b) The absolute min is approximately $(4.9132, -1.814)$ and the absolute max is approximately $(2.0288, 4.820)$.

38. $f'(x) = 2x + e^x = 0$ at approximately $x = -0.3517$.

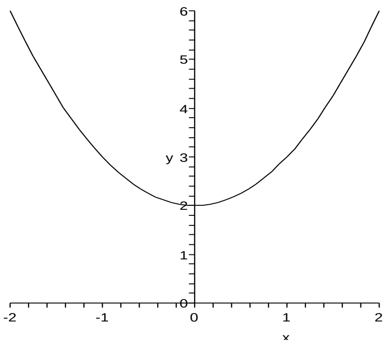
(a) $f(0) = 1$ and $f(1) = 1 + e \approx 3.71828$. $f'(x) \neq 0$ on this interval, so the absolute min is $f(0) = 1$ and the absolute max is $f(1) = 1 + e \approx 3.71828$.

(b) $f(-2) \approx 4.1353$ and $f(2) \approx 11.3891$. $f(-0.3517) = 0.8272$.
The absolute min is approximately $f(-0.3517) = 0.8272$ and the absolute max is approximately $f(2) = 11.3891$.

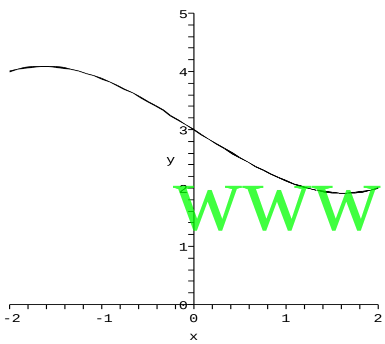
39. On $[-2, 2]$, the absolute maximum is 3 and the absolute minimum doesn't exist.



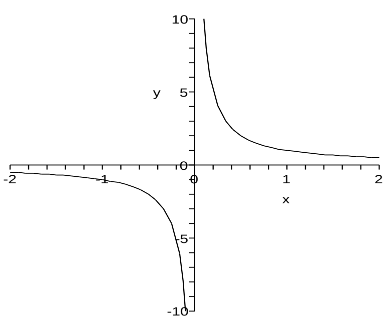
40. On $(-2, 2)$ minimum is 2 and the maximum does not exist. (The maximum would exist at the endpoints which are not included in the interval.)



41. On $(-2, 2)$ the absolute maximum is 4 and the absolute minimum is 2.



42. Absolute extrema do not exist because of the vertical asymptote.



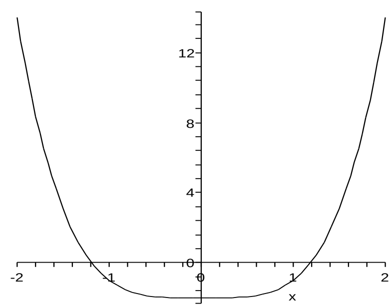
43. $f(x) = x^3 + cx + 1$
$f'(x) = 3x^2 + c$
We know (perhaps from a pre-calculus course) that for any cubic polynomial with positive leading coefficient, when x is large and positive the value of the polynomial is very large and positive, and when x is large and negative, the value of the polynomial is very large and negative.

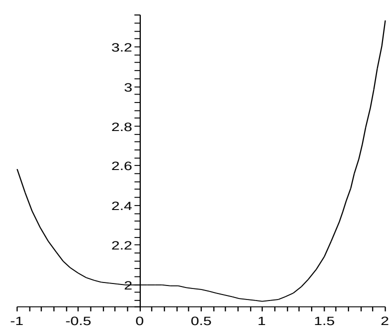
Type 1: $c > 0$. There are no critical numbers. As you move from left to right, the graph of f is always rising.
Type 2: $c < 0$ There are two critical numbers $x = \pm\sqrt{-c/3}$. As you move from left to right, the graph rises until we get to the first critical number, then the graph must fall until we get to the second critical number, and then the graph rises again. So the critical number on the left is a local maximum and the critical number on the right is a local minimum.
Type 3: $c = 0$. There is only one critical number, which is neither a local max nor a local min.

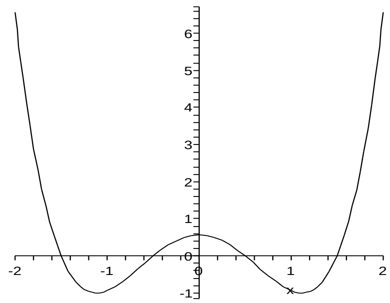
44. The derivative of a fourth-order polynomial is a cubic polynomial. We know that cubic polynomials must have one root, and can have up to three roots. If $p(x)$ is a fourth-order polynomial, we know that

$$\lim_{x\to\infty} p(x) = \lim_{x\to-\infty} p(x) = \infty$$

if the coefficient of x^4 is positive, and is $-\infty$ if the coefficient of x^4 is negative. This guarantees that at least one of the critical numbers will be an extremum.







45. $f(x) = x^3 + bx^2 + cx + d$
$f'(x) = 3x^2 + 2bx + c$
The quadratic formula says that the critical numbers are

$$x = \frac{-2b \pm \sqrt{4b^2 - 12c}}{6} = \frac{-b \pm \sqrt{b^2 - 3c}}{3}.$$

So if $c < 0$, the quantity under the square root is positive and there are two critical numbers. This is like the Type 2 cubics in Exercise 53. We know that as x goes to infinity, the polynomial $x^3 + bx^2 + cx + d$ gets very large and positive, and when x goes to minus infinity, the polynomial is very large but negative. Therefore, the critical number on the left must be a local max, and the critical number on the right must be a local min.

46. $f'(x) = 3x^2 + 2bx + c = 0$ when $x = \dfrac{-2b \pm \sqrt{4b^2 - 12c}}{6}$. Adding these values together yields $-2b/3$.

47. $f(x) = x^4 + cx^2 + 1$
$f'(x) = 4x^3 + 2cx = 2x(2x^2 + c)$
So $x = 0$ is always a critical number.

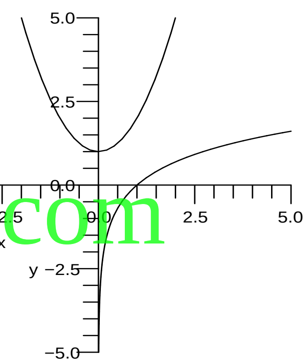
Case 1: $c \geq 0$. The only solution to $2x(2x^2 + c) = 0$ is $x = 0$, so $x = 0$ is the only critical number. This must be a minimum, since we know that the function $x^4 + cx^2 + 1$ is large and positive when $|x|$ is large (so the graph is roughly U-shaped). We could also note that $f(0) = 1$, and 1 is clearly the absolute minimum of this function if $c \geq 0$.

Case 2: $c < 0$. Then there are two other critical numbers $x = \pm\sqrt{-c/2}$. Now $f(0)$ is still equal to 1, but the value of f at both new critical numbers is less than 1. Hence $f(0)$ is a local max, and both new critical numbers are local minimums.

48. $f'(x) = 4x^3 + 3cx^2 = 0$ when $x = 0$ and $x = -3c/4$. Only $x = -3c/4$ will be an extreme point (an absolute minimum). $x = 0$ will be an inflection point.

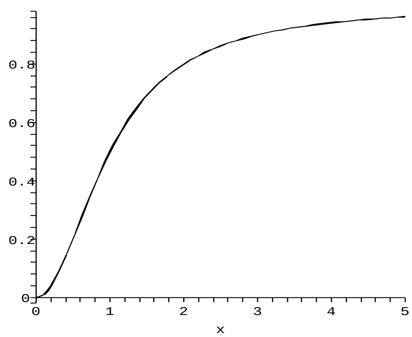
49. Since f is differentiable on (a, b), it is continuous on the same interval. Since f is decreasing at a and increasing at b, f must have a local minimum for some value c, where $a < c < b$. By Fermat's theorem, c is a critical number for f. Since f is differentiable at c, $f'(c)$ exists, and therefore $f'(c) = 0$.

50. Graph of $f(x) = x^2 + 1$ and $g(x) = \ln x$:



$h(x) = f(x) - g(x) = x^2 + 1 - \ln x$
$h'(x) = 2x - 1/x = 0$
$2x^2 = 1$
$x = \pm\sqrt{1/2}$
$x = \sqrt{1/2}$ is min
$f'(x) = 2x$
$g'(x) = 1/x$
$f'\left(\sqrt{1/2}\right) = 2\sqrt{1/2} = \sqrt{2}$
$g'\left(\sqrt{1/2}\right) = \frac{1}{\sqrt{1/2}} = \sqrt{2}$
So the tangents are parallel. If the tangent lines were not parallel, then they would be getting closer together in one direction. Since the tangent lines approximate the curves, this should mean the curves are also getting closer together in that direction.

51. Graph of $f(x) = \dfrac{x^2}{x^2 + 1}$:



$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

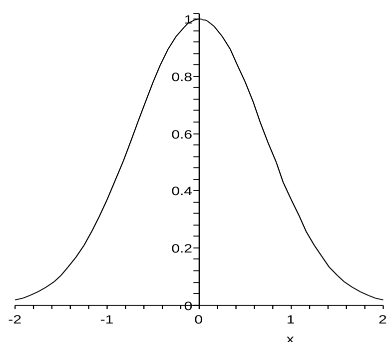
$$f''(x) = \frac{2(x^2+1)^2 - 2x \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4} = \frac{2(x^2+1)\left[(x^2+1) - 4x^2\right]}{(x^2+1)^4} = \frac{2\left[1-3x^2\right]}{(x^2+1)^3}$$

$f''(x) = 0$ for $x = \pm\dfrac{1}{\sqrt{3}}$,

$x = -\dfrac{1}{\sqrt{3}} \notin (0, \infty)$

$x = \dfrac{1}{\sqrt{3}}$ is steepest point.

52. Graph of $f(x) = e^{-x^2}$:



$f(x)$ is steepest where $f'(x) = -2xe^{-x^2}$ is maximum.

$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 0$ when $x = \pm\sqrt{2}/2$. This is where $f(x)$ is steepest.

53. With $t = 90$ and $r = 1/30$, we have $P(n) = \dfrac{3^n}{n!}e^{-3}$. We compute P for the first few values of n:

n	P
0	e^{-3}
1	$3e^{-3}$
2	$4.5e^{-3}$
3	$4.5e^{-3}$
4	$3.375e^{-3}$

Once $n > 3$, the values of P will decrease as n increases. This is due to the fact that to get $P(n+1)$ from $P(n)$, we multiply $P(n)$ by $3/(n+1)$. Since $n > 3$, $3/(n+1) < 1$ and so $P(n+1) < P(n)$. Thus we see from the table that P is maximized at $n = 3$ (it is also maximized at $n = 2$). It makes sense that P would be maximized at $n = 3$ because (90 mins) $\left(\dfrac{1}{30}\text{ goals/min}\right) = 3$ goals.

54. $f(p) = p^m(1-p)^{n-m}$

$f'(p) = mp^{m-1}(1-p)^{n-m}$
$\quad - p^m(n-m)(1-p)^{n-m-1}$

To find the critical numbers, we set $f'(p) = 0$ which gives

$mp^{m-1}(1-p)^{n-m}$
$\quad - p^m(n-m)(1-p)^{n-m-1} = 0$

$mp^{m-1}(1-p)^{n-m}$
$\quad = p^m(n-m)(1-p)^{n-m-1}$

$m(1-p) = p(n-m)$

$m - mp = pn - pm$

$p = m/n$.

Since this is the only critical number, $f(p)$ is continuous, $f(0) = f(1) = 0$ and $f(m/n) > 0$, $p = m/n$ must maximize $f(p)$.

55. $y = x^5 - 4x^3 - x + 10$, $x \in [-2, 2]$

$y' = 5x^4 - 12x^2 - 1$

$x = -1.575$, 1.575 are critical numbers of y. There is a local max at $x = -1.575$, local min at $x = 1.575$.

$x = -1.575$ represents the top and $x = 1.575$ represents the bottom of the roller coaster.

$y''(x) = 20x^3 - 24x = 4x(5x^2 - 6) = 0$

$x = 0$, $\pm\sqrt{6/5}$ are critical numbers of y'. We calculate y' at the critical numbers and at the endpoints $x = \pm 2$:

$y'(0) = -1$

$y'\left(\pm\sqrt{6/5}\right) = -41/5$

$y'(\pm 2) = 31$

So the points where the roller coaster is making the steepest descent are $x = \pm\sqrt{6/5}$, but the steepest part of the roller coast is during the ascents at ± 2.

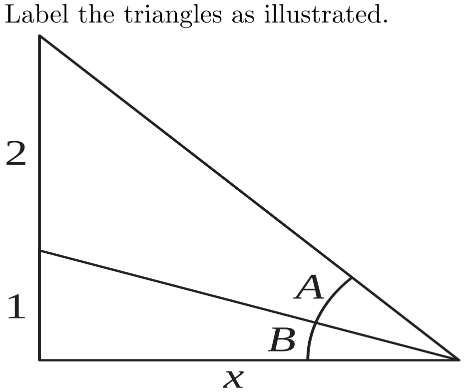
56. To maximize entropy, we find the critical numbers of H.

$H'(x) = -\ln x - 1 + \ln(1-x) + 1 = 0$
where $\ln x = \ln(1-x)$, or where $x = 1-x$. That is $x = 1/2$. This maximizes unpredictablility since for this value, errors and non-errors are equally likely.

57. $W(t) = a \cdot e^{-be^{-t}}$
as $t \to \infty, -be^{-t} \to 0,$ so $W(t) \to a.$
$W'(t) = a \cdot e^{-be^{-t}} \cdot be^{-t}$
as $t \to \infty, be^{-t} \to 0,$ so $W'(t) \to 0.$
$$\begin{aligned} W''(t) &= (a \cdot e^{-be^{-t}} \cdot be^{-t}) \cdot be^{-t} \\ &\quad + (a \cdot e^{-be^{-t}}) \cdot (-be^{-t}) \\ &= a \cdot e^{-be^{-t}} \cdot be^{-t}\,[be^{-t} - 1] \end{aligned}$$
$W''(t) = 0$ when $be^{-t} = 1$
$e^{-t} = b^{-1}$
$-t = \ln b^{-1}$
$t = \ln b$
$$\begin{aligned} W'(\ln b) &= a \cdot e^{-be^{-\ln b}} \cdot be^{-\ln b} \\ &= a \cdot e^{-b\left(\frac{1}{b}\right)} \cdot b \cdot \tfrac{1}{b} = ae^{-1} \end{aligned}$$
Maximum growth rate is ae^{-1} when $t = \ln b$.

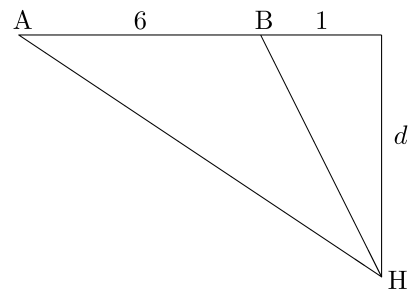
58. $R'([S]) = \dfrac{(K_m + [S])R_m - [S]R_m}{(K_m + [S])^2} \neq 0.$ The function doesn't have a true maximum, but $\lim\limits_{[S]\to\infty} R = R_m$. The rate of reaction approaches R_m but never reaches it.

59. Label the triangles as illustrated.



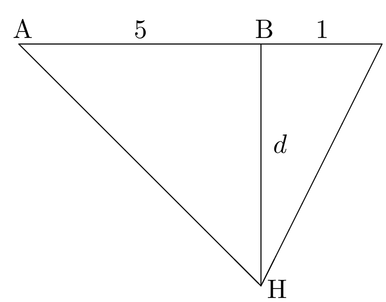
$\tan(A + B) = 3/x$
$\quad A + B = \tan^{-1}(3/x)$
$\tan B = 1/x$
$\quad B = \tan^{-1}(1/x)$
Therefore,
$A = (A + B) - B$
$A = \tan^{-1}(3/x) - \tan^{-1}(1/x)$
$$\begin{aligned} \frac{dA}{dx} &= \frac{-3/x^2}{1 + (3/x)^2} - \frac{-1/x^2}{1 + (1/x)^2} \\ &= \frac{1}{x^2 + 1} - \frac{3}{x^2 + 9} \end{aligned}$$
The maximum viewing angle will occur at a critical value.
$\dfrac{dA}{dx} = 0$
$\dfrac{1}{x^2 + 1} = \dfrac{3}{x^2 + 9}$
$x^2 + 9 = 3x^2 + 3$
$2x^2 = 6$
$x^2 = 3$
$x = \sqrt{3}$ ft ≈ 1.73 ft
This is a maximum because when x is large and when x is a little bigger than 0, the angle is small.

60. (a) For the hockey player, $m\angle$AHB is the shooting angle θ.



Therefore,
$\theta = \tan^{-1}\left(\dfrac{7}{d}\right) - \tan^{-1}\left(\dfrac{1}{d}\right)$
Hence,
$$\begin{aligned} \theta' &= \frac{1}{1 + \left(\frac{49}{d^2}\right)}\left(\frac{-7}{d^2}\right) - \frac{1}{1 + \left(\frac{1}{d^2}\right)}\left(\frac{-1}{d^2}\right) \\ &= \frac{-7}{d^2 + 49} + \frac{1}{d^2 + 1} \end{aligned}$$
To get the maximum angle,
$\theta' = \dfrac{-7}{d^2 + 49} + \dfrac{1}{d^2 + 1} = 0$
$-7d^2 - 7 + d^2 + 49 = 0$
$6d^2 = 42$
$d = \sqrt{7}$

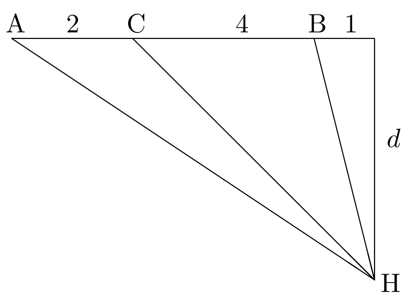
(b) For the hockey player, $m\angle$AHB is the shooting angle θ.



Therefore,
$\theta = \tan^{-1}\left(\frac{5}{d}\right) + \tan^{-1}\left(\frac{1}{d}\right)$
Hence,
$\theta' = \frac{1}{1+\left(\frac{25}{d^2}\right)}\left(\frac{-5}{d^2}\right) + \frac{1}{1+\left(\frac{1}{d^2}\right)}\left(\frac{-1}{d^2}\right)$
$= -\frac{5}{d^2+25} - \frac{1}{d^2+1}$

The function is decreasing as the derivative is negative. Hence the angle is maximum when θ is minimum $= 0$.

(c) For the hockey player, $m\angle\text{AHC}$ is the shooting angle, θ.

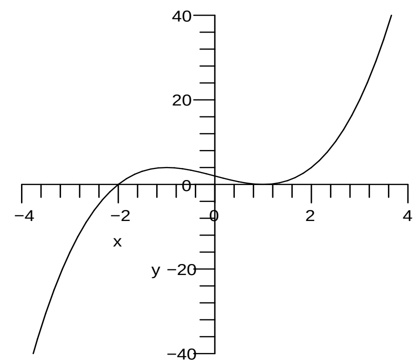


Therefore,
$\theta = \tan^{-1}\left(\frac{7}{d}\right) - \tan^{-1}\left(\frac{5}{d}\right)$
Hence,
$\theta' = \frac{1}{1+\left(\frac{49}{d^2}\right)}\left(\frac{-7}{d^2}\right) - \frac{1}{1+\left(\frac{25}{d^2}\right)}\left(\frac{-5}{d^2}\right)$
$= -\frac{7}{d^2+49} + \frac{5}{d^2+25}$

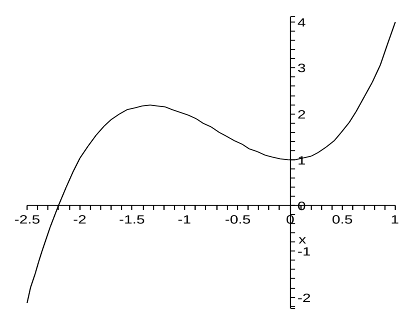
To get the maximum angle,
$\theta' = -\frac{7}{d^2+49} + \frac{5}{d^2+25} = 0$
$-7d^2 - 175 + 5d^2 + 245 = 0$
$2d^2 = 70$
$d = \sqrt{35}$

3.4 Increasing and Decreasing Functions

1. $y = x^3 - 3x + 2$
$y' = 3x^2 - 3 = 3(x^2 - 1)$
$= 3(x+1)(x-1)$
$x = \pm 1$ are critical numbers.
$(x+1) > 0$ on $(-1, \infty)$, $(x+1) < 0$ on $(-\infty, -1)$
$(x-1) > 0$ on $(1, \infty)$, $(x-1) < 0$ on $(-\infty, -1)$
$3(x+1)(x-1) > 0$ on $(1, \infty) \cup (-\infty, -1)$ so y is increasing on $(1, \infty)$ and on $(-\infty, -1)$
$3(x+1)(x-1) < 0$ on $(-1, 1)$, so y is decreasing on $(-1, 1)$.
$y'' = 6x$
$y'' = -6 < 0$ at $x = -1$
Hence the function is a local maximum at $x = -1$.
$y'' = 6 > 0$ at $x = 1$. Hence $y(1) = 0$ is a local minimum.

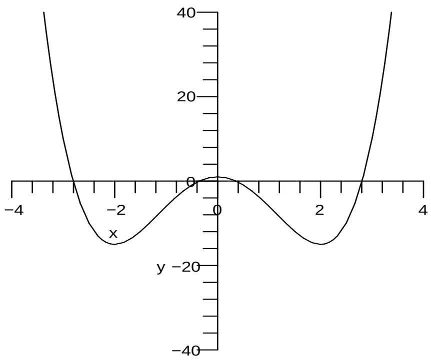


2. $y = x^3 + 2x^2 + 1$
$y' = 3x^2 + 4x = x(3x+4)$
The function is increasing when $x < -\frac{4}{3}$, decreasing when $-\frac{4}{3} < x < 0$, and increasing when $x > 0$.
$y'' = 6x + 4$
$y'' = -12 < 0$ at $x = -\frac{4}{3}$
Hence $f(-\frac{4}{3})$ is a local maximum at $x = -\frac{4}{3}$.
$y'' = 4 > 0$ at $x = 0$
Hence $y(0)$ is a local minimum at $x = 0$.

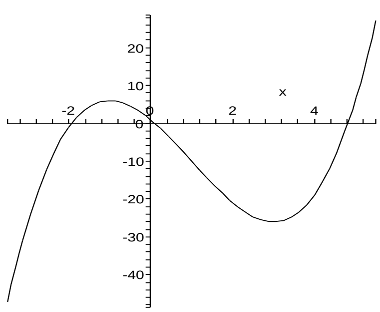


3. $y = x^4 - 8x^2 + 1$
$y' = 4x^3 - 16x = 4x(x^2 - 4)$
$= 4x(x-2)(x+2)$
$x = 0, 2, -2$ are critical numbers.
$4x > 0$ on $(0, \infty)$, $4x < 0$ on $(-\infty, 0)$
$(x-2) > 0$ on $(2, \infty)$, $(x-2) < 0$ on $(-\infty, 2)$
$(x+2) > 0$ on $(-2, \infty)$, $(x+2) < 0$ on $(-\infty, -2)$
$4(x-2)(x+2) > 0$ on $(-2, 0) \cup (2, \infty)$, so the function is increasing on $(-2, 0)$ and on $(2, \infty)$.

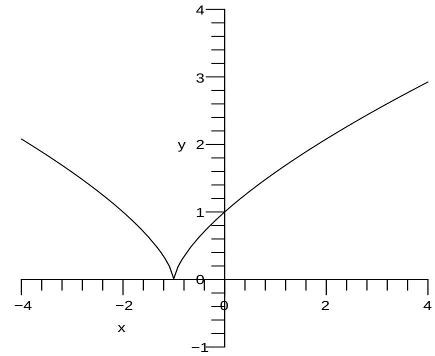
$4(x-2)(x+2) < 0$ on $(-\infty,\,-2)\cup(0,\,2)$, so y is decreasing on $(-\infty,\,-2)$ and on $(0,\,2)$.
$y'' = 12x^2 - 16$
At $x = 0$, $y'' < 0$. Hence $y(0)$ is a local maximum at $x = 0$.
$y'' = 12(\pm 2)^2 - 16 > 0$ at $x = \pm 2$. Hence $y(\pm 2)$ are local minima at $x = \pm 2$.



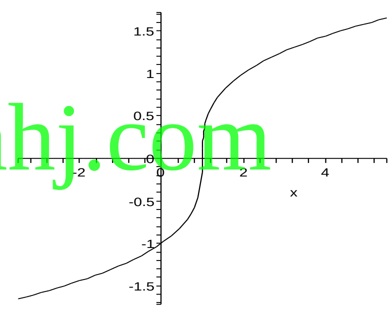
4. $y = x^3 - 3x^2 - 9x + 1$
$y' = 3x^2 - 6x - 9 = 3(x-3)(x+1)$.
The function is increasing when $x < -1$, decreasing when $-1 < x < 3$, and increasing when $x > 3$.
$y'' = 6x - 6$
$y'' = -12 < 0$ at $x = -1$. Hence the function is a local maximum at $x = -1$.
$y'' = 12 > 0$ at $x = 3$. Hence the function is a local minimum at $x = 3$.



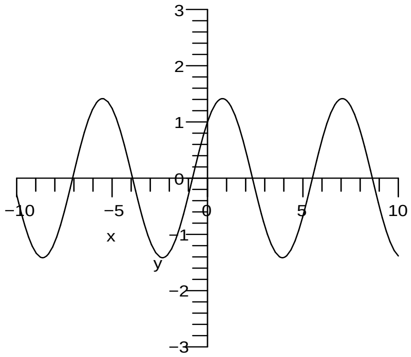
5. $y = (x+1)^{2/3}$
$y' = \frac{2}{3}(x+1)^{-1/3} = \frac{2}{3\sqrt[3]{x+1}}$
y' is not defined for $x = -1$
$\frac{2}{3\sqrt[3]{x+1}} > 0$ on $(-1, \infty)$, y is increasing
$\frac{2}{3\sqrt[3]{x+1}} < 0$ on $(-\infty, -1)$, y is decreasing
The graph has minimum at $x = -1$.



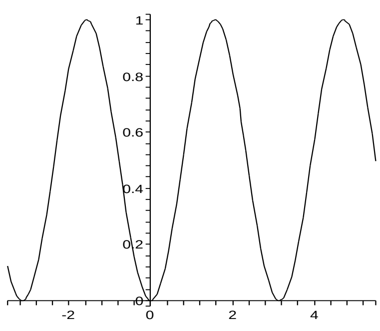
6. $y = (x-1)^{1/3}$
$y' = \frac{1}{3}(x-1)^{-2/3}$.
The function is increasing for all x. The slope approaches vertical as x approaches 1.
The graph has no extrema.



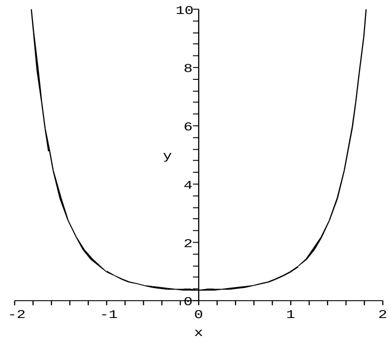
7. $y = \sin x + \cos x$
$y' = \cos x - \sin x = 0$
$\cos x = \sin x$
$x = \pi/4,\ 5\pi/4,\ 9\pi/4$, etc. $\cos x - \sin x > 0$ on $(-3\pi/4, \pi/4) \cup (5\pi/4, 9\pi/4) \cup \ldots$
$\cos x - \sin x < 0$ on $(\pi/4, 5\pi/4) \cup (9\pi/4, 13\pi/4) \cup \ldots$
So $y = \sin x + \cos x$ is decreasing on $(\pi/4, 5\pi/4)\,, (9\pi/4, 13\pi/4)$, etc., and is increasing on $(-3\pi/4, \pi/4)\,, (5\pi/4, 9\pi/4)$, etc.
$y'' = -\sin x - \cos x$
$y'' = -\dfrac{2}{\sqrt{2}} < 0$ at $x = \pi/4$, $x = 9\pi/4$, etc.
Hence the function is local maximum at $x = \pi/4$, $x = 9\pi/4$, etc.
$y'' = \sqrt{2} > 0$ at $x = 5\pi/4$, $x = 13\pi/4$ etc.
Hence the function is local minimum at $x = 5\pi/4$, $x = 13\pi/4$ etc.



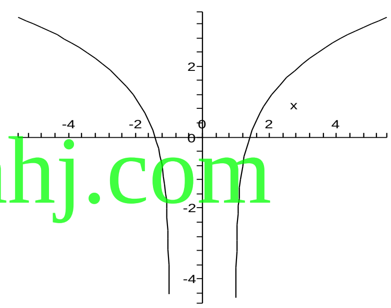
8. $y = \sin^2 x$
$y' = 2 \sin x \cos x$.
The function is increasing for $0 < x < \frac{\pi}{2}$, and decreasing for $\frac{\pi}{2} < x < \pi$, and this pattern repeats with period π.
$y'' = 2 \cos 2x$
$y'' = -2 < 0$ at $x = \pi/2$, $x = 3\pi/2$, etc.
Hence the function is local maximum at $x = \pi/2$, $x = 3\pi/2$, etc.
$y'' = \sqrt{2} > 0$ at $x = 0$, $x = \pi$, etc.
Hence the function is a local minimum $x = 0$, $x = \pi$, etc.



9. $y = e^{x^2-1}$
$y' = e^{x^2-1} \cdot 2x = 2xe^{x^2-1}$
$x = 0$
$2xe^{x^2-1} > 0$ on $(0, \infty)$, y is increasing
$2xe^{x^2-1} < 0$ on $(-\infty, 0)$, y is decreasing
$y'' = 2e^{x^2-1} \left[2x^2 + 1\right]$
$y'' = 0.736 > 0$ at $x = 0$. Hence the function is a local minimum at $x = 0$.



10. $y = \ln(x^2 - 1)$
$y' = \dfrac{2x}{x^2 - 1}$.
The function is defined for $|x| > 1$. The function is decreasing for $x < -1$ and increasing for $x > 1$.
The graph has no extrema.



11. $y = x^4 + 4x^3 - 2$
$y' = 4x^3 + 12x^2 = 4x^2(x + 3)$
Critical numbers are $x = 0$, $x = -3$.
$4x^2(x + 3) > 0$ on $(-3, 0) \cup (0, \infty)$
$4x^2(x + 3) < 0$ on $(-\infty, -3)$
Hence $x = -3$ is a local minimum and $x = 0$ is not an extremum.

12. $y = x^5 - 5x^2 + 1$
$y' = 5x^4 - 10x = 5x(x^3 - 2)$.
At $x = 0$ the slope changes from positive to negative indicating a local maximum. At $x = \sqrt[3]{2}$ the slope changes from negative to positive indicating a local minimum.

13. $y = xe^{-2x}$
$y' = 1 \cdot e^{-2x} + x \cdot e^{-2x}(-2)$
$\quad = e^{-2x} - 2xe^{-2x}$
$\quad = e^{-2x}(1 - 2x)$
$x = \frac{1}{2}$
$e^{-2x}(1 - 2x) > 0$ on $(-\infty, 1/2)$
$e^{-2x}(1 - 2x) < 0$ on $(1/2, \infty)$
So $y = xe^{-2x}$ has a local maximum at $x = 1/2$.

14. $y = x^2 e^{-x}$
$y' = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2 - x)$.
At $x = 0$ the slope changes from negative to positive indicating a local minimum. At $x = 2$ the slope changes from positive to negative indicating a local maximum.

15. $y = \tan^{-1}(x^2)$
$y' = \dfrac{2x}{1 + x^4}$
Critical number is $x = 0$.
$\dfrac{2x}{1 + x^4} > 0$ for $x > 0$
$\dfrac{2x}{1 + x^4} < 0$ for $x < 0$. Hence $x = 0$ is a local minimum.

16. $y = \sin^{-1}\left(1 - \frac{1}{x^2}\right)$
$y' = \dfrac{2}{x^3} \cdot \dfrac{1}{\sqrt{1 - (1 - \frac{1}{x^2})^2}}$.
The derivative is never 0 and is defined where the function is defined, so there are no critical points.

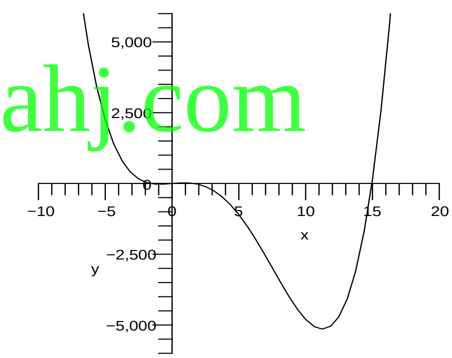
17. $y = \dfrac{x}{1 + x^3}$ Note that the function is not defined for $x = -1$.
$$y' = \frac{1(1 + x^3) - x(3x^2)}{(1 + x^3)} = \frac{1 + x^3 - 3x^3}{(1 + x^3)^2} = \frac{1 - 2x^3}{(1 + x^3)^2}$$
Critical number is $x = \sqrt[3]{1/2}$
$y' > 0$ on $(-\infty, -1) \cup (-1, -\sqrt[3]{1/2})$
$y' < 0$ on $(\sqrt[3]{1/2}, \infty)$
Hence $x = \sqrt[3]{1/2}$ is a local max.

18. $y = \dfrac{x}{1 + x^4}$
$y' = \dfrac{(1 + x^4) - 4x^4}{(1 + x^4)^2} = \dfrac{1 - 3x^4}{(1 + x^4)^2}$.
At $x = -\sqrt[4]{1/3}$ the slope changes from negative to positive indicating a local minimum. At $x = \sqrt[4]{1/3}$ the slope changes from positive to negative incicating a local maximum.

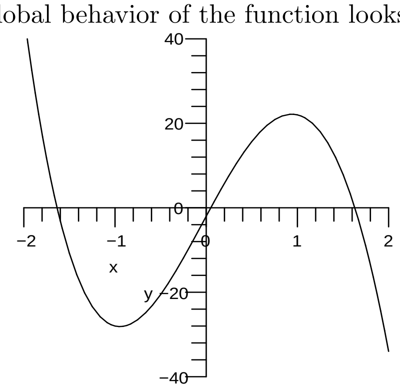
19. $y = \sqrt{x^3 + 3x^2} = (x^3 + 3x^2)^{1/2}$
Domain is all $x \geq -3$.
$$y' = \frac{1}{2}(x^3 + 3x^2)^{-1/2}(3x^2 + 6x) = \frac{3x^2 + 6x}{2\sqrt{x^3 + 3x^2}} = \frac{3x(x + 2)}{2\sqrt{x^3 + 3x^2}}$$
$x = 0, -2, -3$ are critical numbers.
y' undefined at $x = 0, -3$
$y' > 0$ on $(-3, -2) \cup (0, \infty)$
$y' < 0$ on $(-2, 0)$
So $y = \sqrt{x^3 + 3x^2}$ has local max at $x = -2$, local min at $x = 0, -3$.

20. $y = x^{4/3} + 4x^{1/3}$
$y' = \dfrac{4}{3}x^{1/3} + \dfrac{4}{3x^{2/3}} = \dfrac{4}{3} \cdot \dfrac{x + 1}{x^{2/3}}$.
At $x = -1$ the slope changes from negative to positive indicating a local minimum. At $x = 0$ the slope is vertical and is positive on positive side and negative on negative side, so this is neither a minimum nor a maximum.

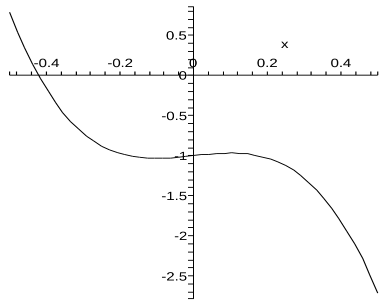
21. $y' = 4x^3 - 45x^2 - 4x + 40$
Local minima at $x = -0.9474, 11.2599$; local max at 0.9374.
Local behavior near $x = 0$ looks like



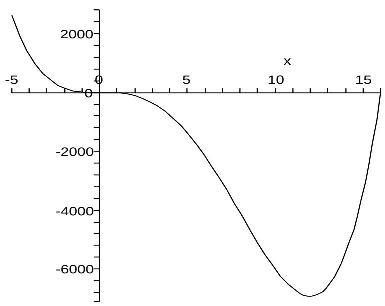
Global behavior of the function looks like



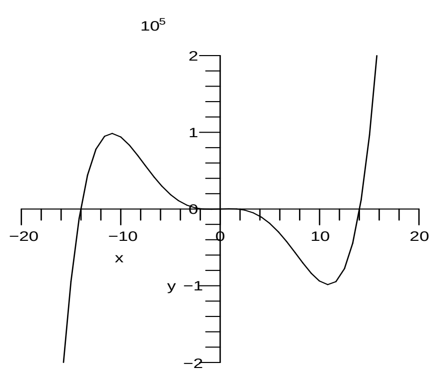
22. $y' = 4x^3 - 48x^2 - 0.2x + 0.5 = 0$ at approximately $x = -0.1037$ (local minimum), $x = 0.1004$ (local maximum), and $x = 12.003$ (local minimum).
Local behavior near $x = 0$ looks like



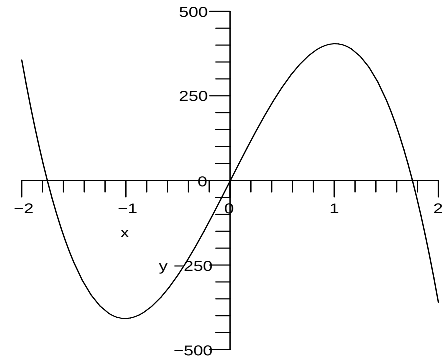
Global behavior of the function looks like



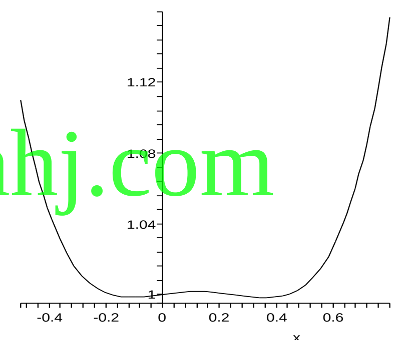
23. $y' = 5x^4 - 600x + 605$
Local minima at $x = -1.0084, 10.9079$; local maxima at $x = -10.9079, 1.0084$.
Local behavior near $x = 0$ looks like



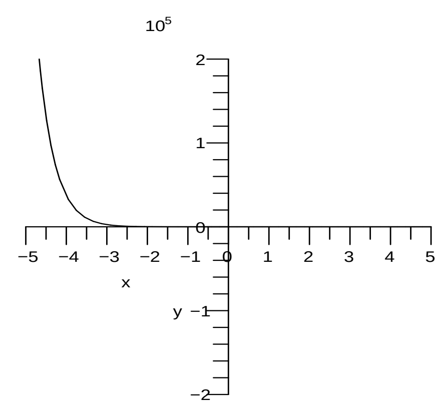
Global behavior of the function looks like



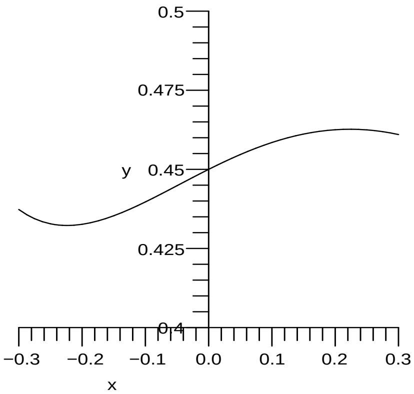
24. $y' = 4x^3 - 1.5x^2 - 0.04x + 0.02 = 0$ at approximately $x = -0.1121$ (local minimum), $x = 0.1223$ (local maximum), and $x = 0.3648$ (local minimum).



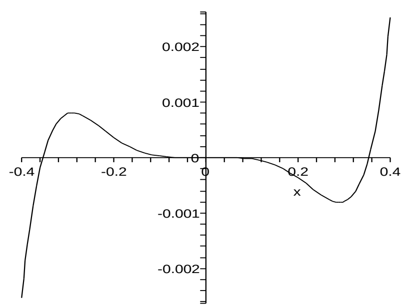
25. $y' = (2x + 1)e^{-2x} + (x^2 + x + 0.45)(-2)e^{-2x}$
Local min at $x = -0.2236$; local max at $x = 0.2236$.
Local behavior near $x = 0$ looks like



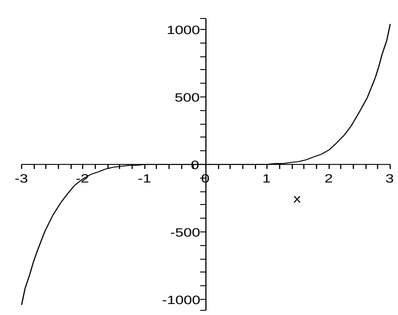
Global behavior of the function looks like



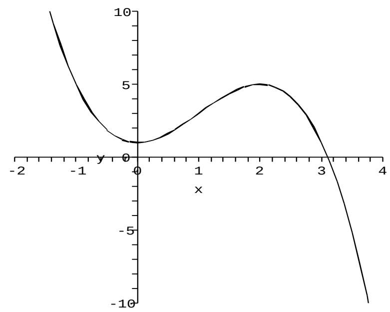
26. $y' = 5x^4 \ln(8x^2) + x^5 \dfrac{16x}{8x^2}$
$= x^4(5\ln(8x^2) + 2) = 0$ at approximately $x = \pm 0.2895$ (a local maximum and local minimum). The derivative and the function are undefined at $x = 0$, but the slope is negative on both sides (neither a minimum nor a maximum).
Locally, near $x = \pm 0.2895$, the function looks like



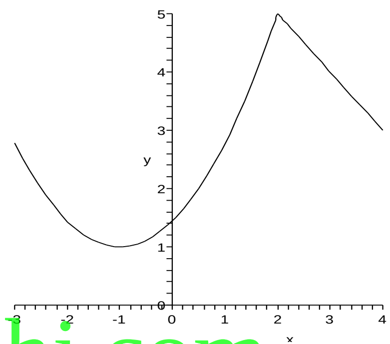
Globally, the function looks like a quintic



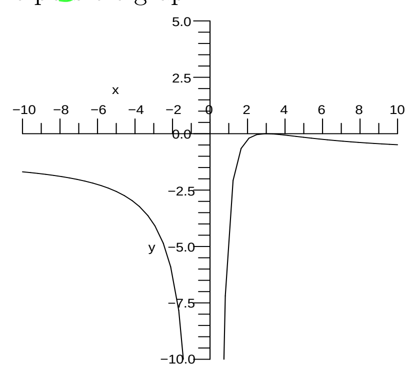
27. One possible graph:



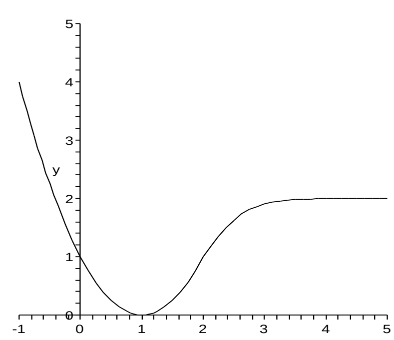
28. One possible graph:



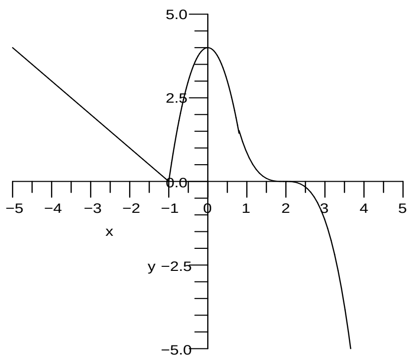
29. One possible graph:



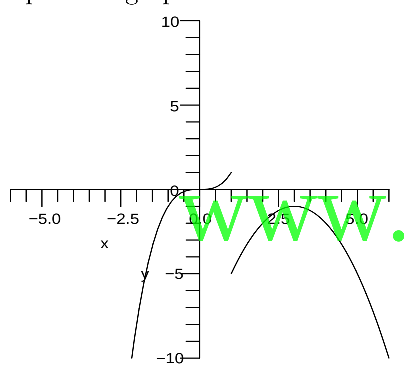
30. One possible graph:



31. One possible graph:



32. One possible graph:



33. $y = \dfrac{x}{x^2-1}$

$$y' = \frac{x^2-1-x(2x)}{(x^2-1)^2}$$
$$= -\frac{x^2+1}{(x^2-1)^2}$$

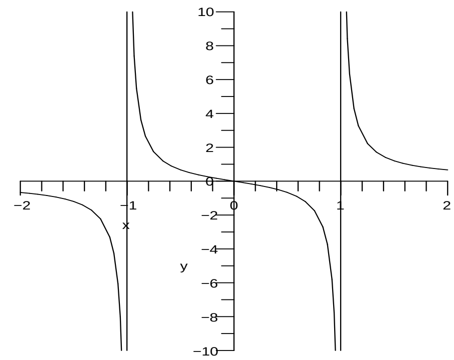
There are no values of x for which $y' = 0$. There are no critical points, because the values for which y' does not exist (that is, $x = \pm 1$) are not in the domain.

There are vertical asymptotes at $x = \pm 1$, and a horizontal asymptote at $y = 0$. This can be verified by calculating the following limits:

$\lim_{x\to\pm\infty} \dfrac{x}{x^2-1} = 0$

$\displaystyle\lim_{x\to -1} \frac{x}{x^2-1} = \infty$

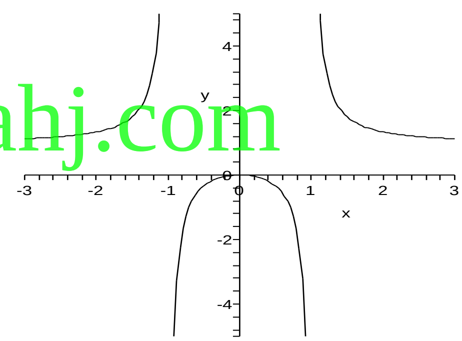
$\displaystyle\lim_{x\to 1} \frac{x}{x^2-1} = -\infty$



34. $y = \dfrac{x^2}{x^2-1}$ has vertical asymptotes at $x = \pm 1$ and horizontal asymptote $y = 1$.

$y' = \dfrac{(x^2-1)2x - 2x(x^2)}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$.

At $x = 0$ the slope changes from positive to negative indicating a local maximum.

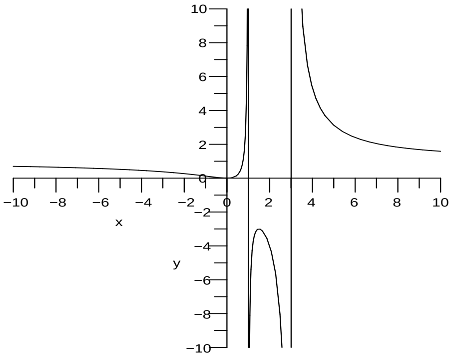


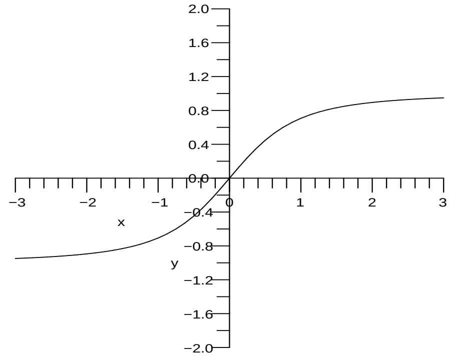
35. $y = \dfrac{x^2}{x^2-4x+3} = \dfrac{x^2}{(x-1)(x-3)}$

Vertical asymptotes $x = 1$, $x = 3$. When $|x|$ is large, the function approaches the value 1, so $y = 1$ is a horizontal asymptote.

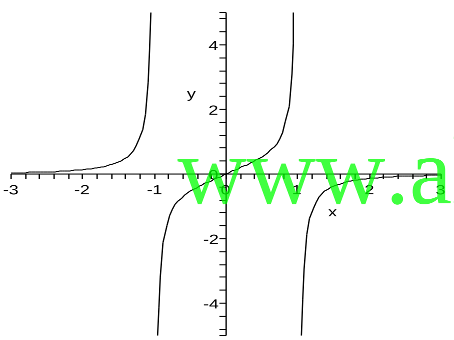
$$y' = \frac{2x(x^2-4x+3) - x^2(2x-4)}{(x^2-4x+3)^2}$$
$$= \frac{2x^3 - 8x^2 + 6x - 2x^3 + 4x^2}{(x^2-4x+3)^2}$$
$$= \frac{-4x^2+6x}{(x^2-4x+3)^2}$$
$$= \frac{2x(-2x+3)}{(x^2-4x+3)^2}$$
$$= \frac{2x(-2x+3)}{[(x-3)(x-1)]^2}$$

Critical numbers are $x = 0$ (local min) and $x = 3/2$ (local max).



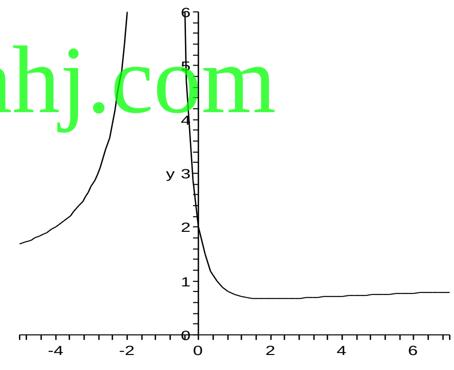


36. $y = \dfrac{x}{1-x^4}$ has vertical asymptotes at $x = \pm 1$ and horizontal asymptote $y = 0$.
$y' = \dfrac{(1-x^4)+4x^4}{(1-x^4)^2} = \dfrac{1+3x^4}{(1-x^4)^2} \neq 0$ for any x and is defined where the function is defined.

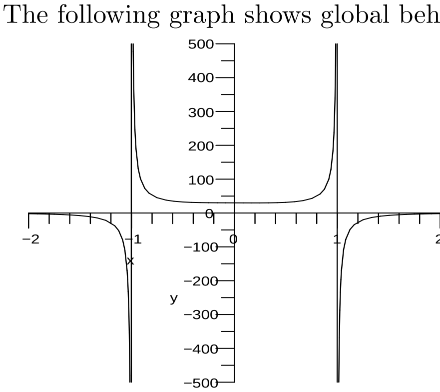


37. $y = \dfrac{x}{\sqrt{x^2+1}}$
$y' = \dfrac{\sqrt{x^2+1} - x^2/\sqrt{x^2+1}}{x^2+1}$
$= \dfrac{1}{(x^2+1)^{3/2}}$
The derivative is never zero, so there are no critical points. To verify that there are horizontal asymptotes at $y = \pm 1$: $y = \dfrac{x}{\sqrt{x^2+1}}$
$= \dfrac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}$
$= \dfrac{x}{|x|\sqrt{1+\frac{1}{x^2}}}$
Thus,
$\lim_{x\to\infty} \dfrac{x}{|x|\sqrt{1+\frac{1}{x^2}}} = 1$
$\lim\limits_{x\to-\infty} \dfrac{x}{|x|\sqrt{1+\frac{1}{x^2}}} = -1$

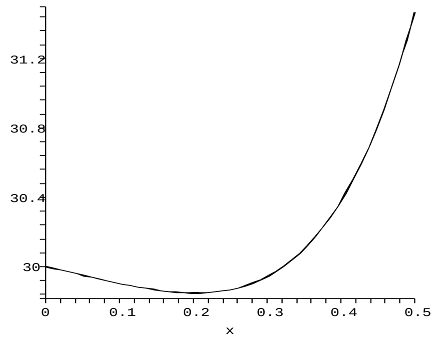
38. $y = \dfrac{x^2+2}{(x+1)^2}$ has a vertical asymptote at $x = -1$, and a horizontal asymptote at $y = 1$.
$y' = \dfrac{2x(x+1)^2 - (x^2+2)2(x+1)}{(x+1)^4}$
$= \dfrac{2(x-2)(x+1)}{(x+1)^4}$
$x = 2$ is the only critical number. Since $f'(0) < 0$ and $f'(3) > 0$, we see that $f(2)$ is a local minimum.

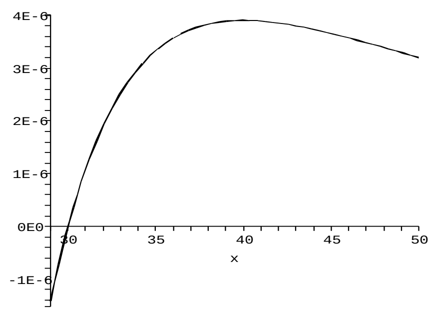


39. The derivative is
$y' = \dfrac{-3x^4+120x^3-1}{(x^4-1)^2}$.
We estimate the critical numbers to be approximately 0.2031 and 39.999.
The following graph shows global behavior:

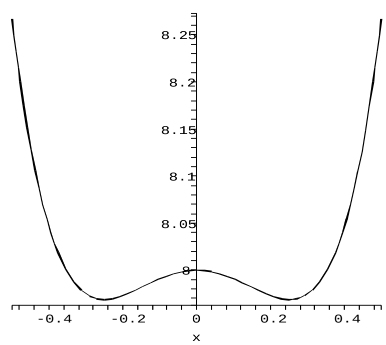


The following graphs show local behavior:

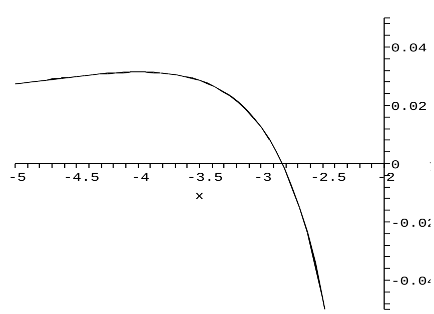


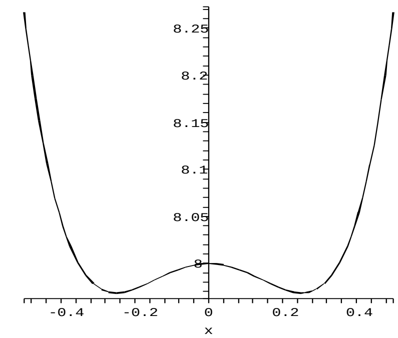


40. The derivative is
$y' = \dfrac{-2x^5 + 32x^3 - 2x}{(x^4 - 1)^2}$.
We estimate the critical numbers to be approximately ± 0.251, ± 3.992 and $x = 0$.
The following graph shows global behavior:

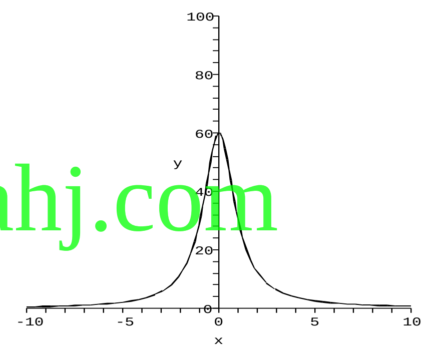


The following graphs show local behavior:

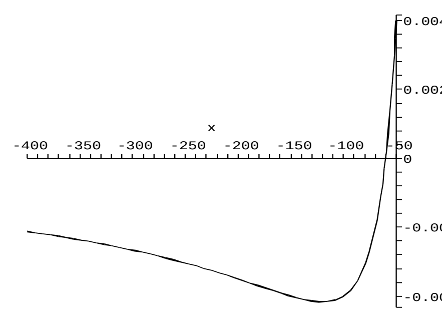


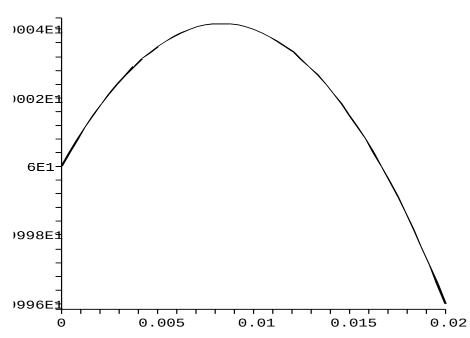


41. The derivative is $y' = \dfrac{-x^2 - 120x + 1}{(x^2 + 1)^2}$.
We estimate the critical numbers to be approximately 0.008 and -120.008.
The following graph shows global behavior:

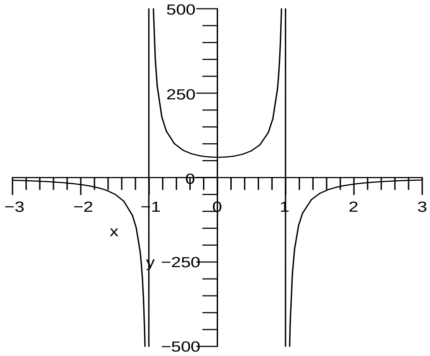


The following graphs show local behavior:

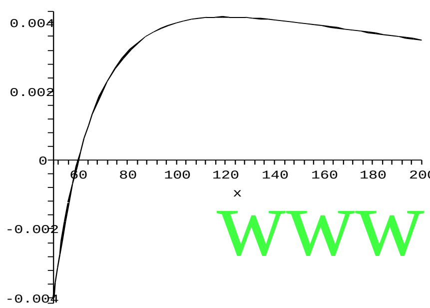


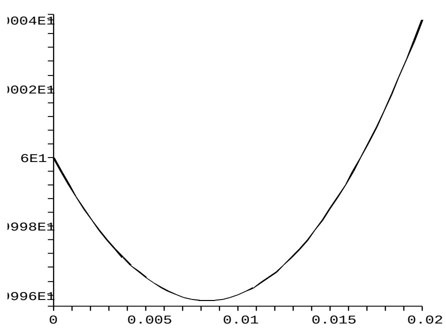


42. The derivative is $y' = \dfrac{-x^2+120x-1}{(x^2-1)^2}$.
We estimate the critical numbers to be approximately 0.008 and 119.992.
The following graph shows global behavior:



The following graphs show local behavior:





43. Let $f(x) = 3 + e^{-x}$; then $f(0) = 4$, $f'(x) = -e^{-x} < 0$, so f is decreasing. But $f(x) = 3 + e^{-x} = 0$ has no solution.

44. Let y_1 and y_2 be two points in the domain of f^{-1} with $y_1 < y_2$. Let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. We want to show $x_1 < x_2$. Suppose not. Then $x_2 \le x_1$. But then, since f is increasing, $f(x_2) \le f(x_1)$. That is $y_2 \le y_1$, which contradicts our choice of y_1 and y_2.

45. The domain of $\sin^{-1} x$ is the interval $[-1, 1]$. The function is increasing on the entire domain.

46. $\sin^{-1}\left(\dfrac{2}{\pi}\tan^{-1} x\right)$ is defined for all x. The derivative,
$$\frac{2}{\pi(1+x^2)\sqrt{1-(\frac{2}{\pi}\tan^{-1} x)^2}} > 0$$
for all x. The function is increasing everywhere.

47. TRUE. If $x_1 < x_2$, then $g(x_1) < g(x_2)$ since g is increasing, and then $f(g(x_1)) < f(g(x_2))$ since f is increasing.

48. We can say that $g(1) < g(4)$ and $g(f(1)) < g(f(4))$, but it is not possible to determine the maximum and minimum values without more information.

49.
$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\ &= \lim_{x\to 0} \frac{f(x)}{x} \\ &= \lim_{x\to 0} \left[1 + 2x\sin\left(\frac{1}{x}\right)\right] = 1 \end{aligned}$$
For $x \ne 0$,
$$\begin{aligned} &f'(x) \\ &= 1 + 2\left[2x\sin\left(\frac{1}{x}\right) + x^2\left(\frac{-1}{x^2}\right)\cos\left(\frac{1}{x}\right)\right] \\ &= 1 + 4x\sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right) \end{aligned}$$
For values of x close to the origin, the middle term of the derivative is small, and since the last term $-2\cos(1/x)$ reaches its minimum value of -2 in every neighborhood of the origin, f' has negative values on every neighborhood of the origin. Thus, f is not increasing on any neighborhood of the origin. This conclusion does not contradict Theorem 4.1 because the theorem states that if a function's derivative is positive for all values in an interval, then it is increasing in that interval. In this example, the derivative is not positive throughout any interval containing the origin.

50. We have $f'(x) = 3x^2$, so $f'(x) > 0$ for all $x \ne 0$, but $f'(0) = 0$. Since $f'(x) > 0$ for all $x \ne 0$, we know $f(x)$ is increasing on any interval not containing 0. We know that if $x_1 < 0$ then $x_1^3 < 0$ and if $x_2 > 0$ then $x_2^3 > 0$. If $x_1 < 0$ and $x_2 = 0$ then $x_1^3 < 0^3 = 0$, so $f(x)$ is increasing on intervals of the form $(x_1, 0)$. Similarly, $f(x)$ is increasing on intervals of the form $(0, x_2)$. Finally, on intervals of the form (x_1, x_2) where $x_1 < 0 < x_2$, we have $x_1^3 < 0 < x_2^3$ so $f(x)$ is again increasing on these intervals. Thus $f(x)$ is increasing on any interval.
This does not contradict Theorem 4.1 because

Theorem 4.1 is not an "if and only if" statement. It says that if $f'(x) > 0$, then f is increasing (on that interval) but it does not say that if $f'(x)$ is not strictly positive that f is not increasing.

51. f is continuous on $[a, b]$, and $c \in (a, b)$ is a critical number.

(i) If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, by Theorem 3.1, f is increasing on (a, c) and decreasing on (c, b), so $f(c) > f(x)$ for all $x \in (a, c)$ and $x \in (c, b)$. Thus $f(c)$ is a local max.

(ii) If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, by Theorem 3.1, f is decreasing on (a, c) and increasing on (c, b). So $f(c) < f(x)$ for all $x \in (a, c)$ and $x \in (c, b)$. Thus $f(c)$ is a local min.

(iii) If $f'(x) > 0$ on (a, c) and (c, b), then $f(c) > f(x)$ for all $x \in (a, c)$ and $f(c) < f(x)$ for all $x \in (c, b)$, so c is not a local extremum. If $f'(x) < 0$ on (a, c) and (c, b), then $f(c) < f(x)$ for all $x \in (a, c)$ and $f(c) > f(x)$ for all $x \in (c, b)$, so c is not a local extremum.

52. If $f(a) = g(a)$ and $f'(x) > g'(x)$ for all $x > a$, then $f(x) > g(x)$ for all $x > a$. Graphically, this makes sense: f and g start at the same place, but f is increasing faster, therefore f should be larger than g for all $x > a$. To prove this, apply the Mean Value Theorem to the function $f(x) - g(x)$.
If $x > a$ then there exists a number c between a and x with
$f'(c) - g'(c) = \dfrac{(f(x) - g(x)) - (f(a) - g(a))}{x - a}$.
Multiply by $(x - a)$ (and recall $f(a) = g(a)$) to get $(x - a)(f'(c) - g'(c)) = f(x) - g(x)$. The lefthand side of this equation is positive, therefore $f(x)$ is greater than $g(x)$.

53. Let $f(x) = 2\sqrt{x}$, $g(x) = 3 - 1/x$.
Then $f(1) = 2\sqrt{1} = 2$, and $g(1) = 3 - 1 = 2$, so $f(1) = g(1)$.
$f'(x) = \dfrac{1}{\sqrt{x}}$, $g'(x) = \dfrac{1}{x^2}$
So $f'(x) > g'(x)$ for all $x > 1$, and
$f(x) = 2\sqrt{x} > 3 - \dfrac{1}{x} = g(x)$ for all $x > 1$.

54. Let $f(x) = x$ and $g(x) = \sin x$.
Then $f(0) = g(0)$. $f'(x) = 1$. $g'(x) = \cos x$.
$\cos x \le 1$ for all x, therefore exercise 52 implies that $x > \sin x$ for all $x > 0$.

55. Let $f(x) = e^x$, $g(x) = x + 1$.
Then $f(0) = e^0 = 1$, $g(0) = 0 + 1 = 1$, so $f(0) = g(0)$.
$f'(x) = e^x$, $g'(x) = 1$
So $f'(x) > g'(x)$ for $x > 0$.
Thus $f(x) = e^x > x + 1 = g(x)$ for $x > 0$.

56. Let $f(x) = x - 1$ and $g(x) = \ln x$.
Then $f(1) = g(1)$. $f'(x) = 1$. $g'(x) = \frac{1}{x}$.
$1/x \le 1$ for all $x > 1$, therefore exercise 52 implies that $x - 1 > \ln x$ for all $x > 1$.

57. $f(x) = x^3 + bx^2 + cx + d$
$f'(x) = 3x^2 + 2bx + c$
$f'(x) \ge 0$ for all x if and only if
$(2b)^2 - 4(3)(c) \le 0$
if and only if $4b^2 \le 12c$
if and only if $b^2 \le 3c$.
Using the quadratic formula, we find
$x^2 = \dfrac{-3b \pm \sqrt{9b^2 - 20c}}{10}$.
Thus, if $9b^2 < 20c$, then the roots are imaginary and so $f'(x) \ge 0$ for all x. If this is not the case, then we need to consider
$x = \pm\sqrt{\dfrac{-3b \pm \sqrt{9b^2 - 20c}}{10}}$.
Now we need the expression inside the square root to be less than or equal to 0, which is the same as requiring the numerator of the expression inside the square root to be less than or equal to 0. So we need both
$-3b < \sqrt{9b^2 - 20c}$ and

$-3b < -\sqrt{9b^2 - 20c}$.

Of course, both are true if and only if the latter is true. In conclusion, $f(x)$ is an increasing function if $9b^2 < 20c$ or $-3b < -\sqrt{9b^2 - 20c}$.

58. TRUE. $(f \circ g)'(c) = f'(g(c))g'(c) = 0$, since c is a critical number of g.

59. $s(t) = \sqrt{t + 4} = (t + 4)^{1/2}$
$s'(t) = \dfrac{1}{2}(t + 4)^{-1/2} = \dfrac{1}{2\sqrt{t + 4}} > 0$
So total sales are always increasing at the rate of $\dfrac{1}{2\sqrt{t + 4}}$ thousand dollars per month.

60. $s'(t) = \dfrac{1}{2\sqrt{t + 4}} > 0$ for all $t > 0$. If s represents the total sales so far, then s cannot decrease. The rate of new sales can decrease, but we cannot lose sales that already have occurred.

61. (a) $\mu'(-10) \approx \dfrac{0.0048 - 0.0043}{-12 - (-8)}$
$= \dfrac{0.0005}{-4}$
$= -0.000125$

(b) $\mu'(-6) \approx \dfrac{0.0048 - 0.0043}{-4 - (-8)}$
$= \dfrac{0.0005}{4}$
$= 0.000125$

Whether the warming of the ice due to skating makes it easier or harder depends on the current temperature of the ice. As seen from these examples, the coefficient of friction μ is decreasing when the temperature is $-10°$ and increasing when the temperature is $-6°$.

62. We find the derivative of $f(t)$:

$$f'(t) = \frac{a^2 + t^2 - t(2t)}{(a^2+t^2)^2} = \frac{a^2 - t^2}{(a^2+t^2)^2}.$$

The denominator is always positive, while the numerator is positive when $a^2 > t^2$, i.e., when $a > t$. We now find the derivative of $\theta(x)$:

$$\theta'(x) = \frac{1}{1 + \left(\frac{29.25}{x}\right)^2}\left(\frac{-29.25}{x^2}\right) - \frac{1}{1 + \left(\frac{10.75}{x}\right)^2}\left(\frac{-10.75}{x^2}\right) = \frac{-29.25}{x^2 + (29.25)^2} + \frac{10.75}{x^2 + (10.75)^2}.$$

We consider each of the two terms of the last line above as instances of $f(t)$, the first as $-f(29.25)$ and the second as $f(10.75)$. Now, for any given x where $x \geq 30$, this x is our a in $f(t)$ and since $a = x$ is greater than 29.25 and greater than 10.75, $f(t)$ is increasing for these two t values and this value of a. Thus $f(29.25) > f(10.75)$. This means that
$\theta'(x) = -f(29.25) + f(10.75) < 0$
(where $a = x$) and so $\theta(x)$ is decreasing for $x \geq 30$. Since $\theta(x)$ is increasing for $x \geq 30$, the announcers would be wrong to suggest that the angle increases by backing up 5 yards when the team is between 50 and 60 feet away from the goal post.

3.5 Concavity and the Second Derivative Test

1. $f'(x) = 3x^2 - 6x + 4$
$f''(x) = 6x - 6 = 6(x-1)$
$f''(x) > 0$ on $(1, \infty)$
$f''(x) < 0$ on $(-\infty, 1)$
So f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.
$x = 1$ is a point of inflection.

2. $f'(x) = 4x^3 - 12x + 2$ and $f''(x) = 12x^2 - 12$. The graph is concave up where $f''(x)$ is positive, and concave down where $f''(x)$ is negative. Concave up for $x < -1$ and $x > 1$, and concave down for $-1 < x < 1$.
$x = -1, 1$ are points of inflection.

3. $f(x) = x + \frac{1}{x} = x + x^{-1}$
$f'(x) = 1 - x^{-2}$
$f''(x) = 2x^{-3}$
$f''(x) > 0$ on $(0, \infty)$
$f''(x) < 0$ on $(-\infty, 0)$
So f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.
$x = 0$ is a point of inflection.

4. $y' = 1 - (1-x)^{-2/3}$ and $y'' = \frac{-2}{3}(1-x)^{-5/3}$. Concave up for $x > 1$ and concave down for $x < 1$.
$x = 1$ is a point of inflection.

5. $f'(x) = \cos x + \sin x$
$f''(x) = -\sin x + \cos x$
$f''(x) < 0$ on $\ldots\left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{9\pi}{4}, \frac{13\pi}{4}\right)\ldots$
$f''(x) > 0$ on $\ldots\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)\ldots$
f is concave down on $\ldots\left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{9\pi}{4}, \frac{13\pi}{4}\right)\ldots$,
concave up on $\ldots\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)\ldots$
$x = k\pi + \frac{\pi}{4}$ are the points of inflection for any interger k.

6. $f'(x) = \dfrac{2x}{1+x^4}$ and $f''(x) = \dfrac{2 - 6x^4}{(1+x^4)^2}$.

Concave up for $-\sqrt[4]{\frac{1}{3}} < x < \sqrt[4]{\frac{1}{3}}$, and concave down for $x < -\sqrt[4]{\frac{1}{3}}$ and $x > \sqrt[4]{\frac{1}{3}}$.

$x = -\sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}$ are the points of inflection.

7. $f'(x) = \dfrac{4}{3}x^{1/3} + \dfrac{4}{3}x^{-2/3}$

$$f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9x^{2/3}}\left(1 - \frac{2}{x}\right)$$

The quantity $\frac{4}{9x^{2/3}}$ is never negative, so the sign of the second derivative is the same as the sign of $1 - \frac{2}{x}$. Hence the function is concave up for $x > 2$ and $x < 0$, and is concave down for $0 < x < 2$.
$x = 0\,2$ are the points of inflection.

8. $f'(x) = e^{-4x} - 4xe^{-4x}$ and $f''(x) = 8e^{-4x}(2x - 1)$.
Concave up for $x > 1/2$, and concave down for $x < 1/2$.
$x = 1/2$ is the point of inflection.

9. $f(x) = x^4 + 4x^3 - 1$
$f'(x) = 4x^3 + 12x^2 = x^2(4x + 12)$
So the critical numbers are $x = 0$ and $x = -3$.
$f''(x) = 12x^2 + 24x$
$f''(0) = 0$ so the second derivative test for $x = 0$ is inconclusive.
$f''(-3) = 36 > 0$ so $x = -3$ is a local minimum.

10. $f(x) = x^4 + 4x^2 + 1$
$f'(x) = 4x^3 + 8x$
So the only critical number is $x = 0$.
$f''(x) = 12x^2 + 8$
$f''(0) = 8 > 0$ so $x = 0$ is a local minimum.

11. $f(x) = xe^{-x}$
$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$
So the only critical number is $x = 1$.
$f''(x) = -e^{-x} - e^{-x} + xe^{-x} = e^{-x}(-2 + x)$
$f''(1) = e^{-1}(-1) < 0$ so $x = 1$ is a local maximum.

12. $f(x) = e^{-x^2}$
$f'(x) = -2xe^{-x^2}$
So the only critical number is $x = 0$.
$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$
$f''(0) = -2 + 0 < 0$ so $x = 0$ is a local maximum.

13. $f(x) = \dfrac{x^2 - 5x + 4}{x}$

$$f'(x) = \frac{(2x - 5)x - (x^2 - 5x + 4)(1)}{x^2} = \frac{x^2 - 4}{x^2}$$

So the critical numbers are $x = \pm 2$.

$$f''(x) = \frac{(2x)(x^2) - (x^2 - 4)(2x)}{x^4} = \frac{8x}{x^4}$$

$f''(2) = 1 > 0$ so $x = 2$ is a local minimum.
$f''(-2) = -1 < 0$ so $x = -2$ is a local maximum.

14. $f(x) = \dfrac{x^2 - 1}{x}$

$$f'(x) = \frac{(2x)(x) - (x^2 - 1)(1)}{x^2} = \frac{x^2 + 1}{x^2}$$

There are no critical numbers and so there are no local extrema.

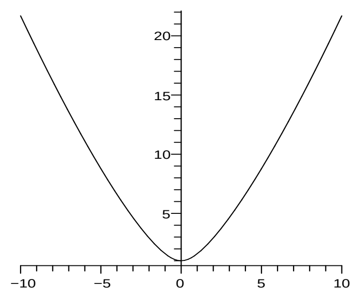
15. $y = (x^2 + 1)^{2/3}$
$y' = \frac{2}{3}(x^2 + 1)^{-1/3}(2x)$
$f'(x) = \dfrac{4x(x^2 + 1)^{-1/3}}{3}$
So the only critical number is $x = 0$.

$$y'' = \frac{4}{3}\left[(x^2 + 1)^{-1/3} + \left(\frac{-2x^2}{3}\right)(x^2 + 1)^{-4/3}\right]$$

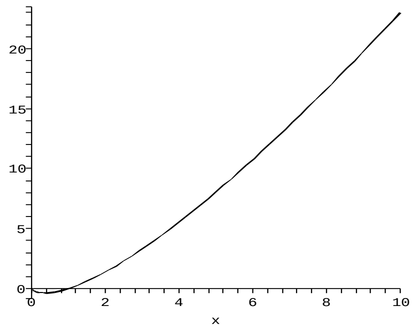
$$= \frac{4}{3}\frac{(x^2 + 1 - \frac{2x^2}{3})}{(x^2 + 1)^{4/3}} = \frac{4}{9}\frac{(3x^2 + 3 - 2x^2)}{(x^2 + 1)^{4/3}}$$

$$= \frac{4}{9}\frac{(x^2 + 3)}{(x^2 + 1)^{4/3}}$$

So the function is concave up everywhere, decreasing for $x < 0$, and increasing for $x > 0$. Also $x = 0$ is a local min.



16. $f(x) = x \ln x$
$f'(x) = \ln x + 1$
So the only critical number is e^{-1}.
$f''(x) = 1/x$
$f''(e^{-1}) = e > 0$ so $f(x)$ has a local minimum at $x = e^{-1}$.
The domain of $f(x)$ is $(0, \infty)$.
$f'(x) < 0$ on $(0, e^{-1})$ so $f(x)$ is decreasing on this interval. $f'(x) > 0$ on (e^{-1}, ∞), so $f(x)$ is increasing on this interval.
$f''(x) > 0$ for all x in the domain of $f(x)$, so $f(x)$ is concave up for all $x > 0$.
Finally, $f(x)$ has a vertical asymptote at $x = 0$ such that $f(x) \to \infty$ as $x \to 0^+$.



17. $f(x) = \dfrac{x^2}{x^2-9}$

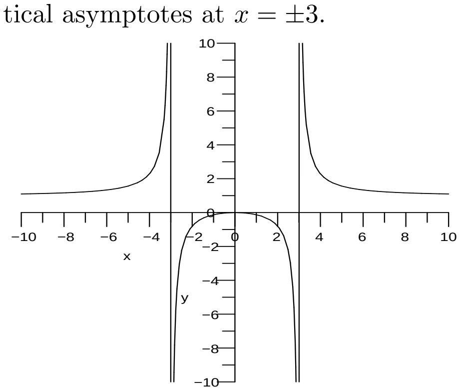
$$f'(x) = \frac{2x(x^2-9) - x^2(2x)}{(x^2-9)^2}$$
$$= \frac{-18x}{(x^2-9)^2}$$
$$= \frac{-18x}{\{(x+3)(x-3)\}^2}$$

$f''(x) =$
$$\frac{-18(x^2-9)^2 + 18x \cdot 2(x^2-9) \cdot 2x}{(x^2-9)^4}$$
$$= \frac{54x^2+162}{(x^2-9)^3}$$
$$= \frac{54(x^2+3)}{(x^2-9)^3}$$

$f'(x) > 0$ on $(-\infty, -3) \cup (-3, 0)$
$f'(x) < 0$ on $(0, 3) \cup (3, \infty)$
$f''(x) > 0$ on $(-\infty, -3) \cup (3, \infty)$
$f''(x) < 0$ on $(-3, 3)$
$f''(0) = \dfrac{162}{(-9)^3}$

f is increasing on $(-\infty, -3) \cup (-3, 0)$, decreasing on $(0, 3) \cup (3, \infty)$, concave up on $(-\infty, -3) \cup (3, \infty)$, concave down on $(-3, 3)$, $x = 0$ is a local max.

f has a horizontal asymptote of $y = 1$ and vertical asymptotes at $x = \pm 3$.



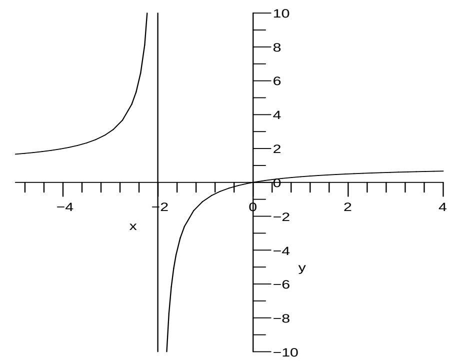
18. $f(x) = \dfrac{x}{x+2}$
The domain of $f(x)$ is $\{x | x \neq -2\}$.

There is a vertical asymptote at $x = -2$ such that $f(x) \to \infty$ as $x \to -2^-$ and $f(x) \to -\infty$ as $x \to -2^+$.

$f'(x) = \dfrac{x+2-x}{(x+2)^2} = \dfrac{2}{(x+2)^2}$

So there are no critical numbers. Furthermore, $f'(x) > 0$ for all $x \neq -2$, so $f(x)$ is increasing everywhere.

$f''(x) = -4(x+2)^{-3}$
$f''(x) > 0$ on $(-\infty, -2)$ (so $f(x)$ is concave up on this interval)
$f''(x) > 0$ on $(-2, \infty)$ (so $f(x)$ is concave down on this interval)



19. $f(x) = \sin x + \cos x$
$f'(x) = \cos x - \sin x$
$f''(x) = -\sin x - \cos x$

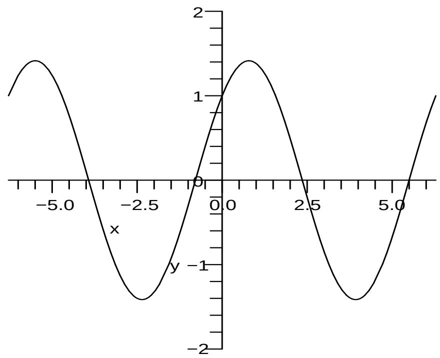
$f'(x) = 0$ when $x = \pi/4 + k\pi$ for all integers k. When k is even, $f''(\pi/4 + k\pi) = -\sqrt{2} < 0$ so $f(x)$ has a local maximum. When k is odd, $f''(\pi/4 + k\pi) = \sqrt{2} > 0$ so $f(x)$ has a local minimum.

$f'(x) < 0$ on the intervals of the form $(\pi/4 + 2k\pi, \pi/4 + (2k+1)\pi)$, so $f(x)$ is decreasing on these intervals.

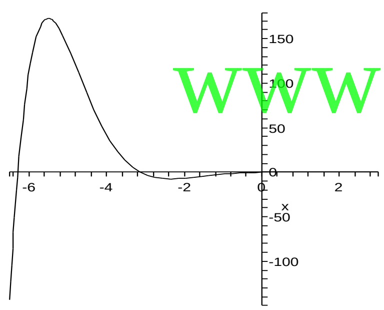
$f'(x) > 0$ on the intervals of the form $(\pi/4 + (2k+1)\pi, \pi/4 + (2k+2)\pi)$, so $f(x)$ is increasing on these intervals.

$f''(x) > 0$ on the intervals of the form $(3\pi/4 + 2k\pi, 3\pi/4 + (2k+1)\pi)$ so $f(x)$ is concave up on these intervals.

$f''(x) < 0$ on the intervals of the form $(3\pi/4 + (2k+1)\pi, 3\pi/4 + (2k+2)\pi)$ so $f(x)$ is concave down on these intervals.



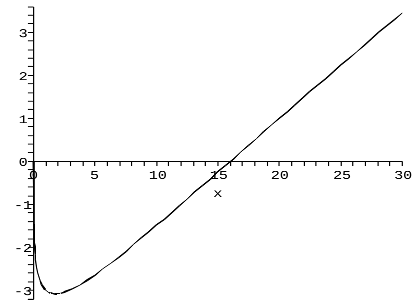
20. $y = e^{-x}\sin x$
$y' = -e^{-x}\sin x + e^{-x}\cos x = 0$ when $x = \pi/4 + k\pi$ for integers k.
$y'' = -2e^{-x}\cos x = 0$ at $\pi/2 + 2k\pi$ for integers k. These are inflection points. The function is concave up for $-\pi/2 < x < \pi/2$ and concave down for $\pi/2 < x < 3\pi/2$, and the pattern repeats with period 2π. The critical values are all extrema, and they alternate between maxima and minima.



21. $f(x) = x^{3/4} - 4x^{1/4}$
Domain of $f(x)$ is $\{x | x \geq 0\}$.

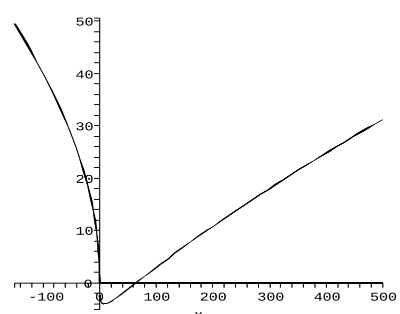
$$f'(x) = \frac{3}{4}x^{-1/4} - x^{-3/4} = \frac{\frac{3}{4}\sqrt{x} - 1}{x^{3/4}}$$

So $x = 0$ and $x = 16/9$ are critical points, but because of the domain we only need to really consider the latter.
$f'(1) = -1/4$ so $f(x)$ is decreasing on $(0, 16/9)$.
$f'(4) = \dfrac{0.5}{4^{3/4}} > 0$ so $f(x)$ is increasing on $(16/9, \infty)$.
Thus $x = 16/9$ is the location of a local minimum for $f(x)$.
$f''(x) = \frac{-3}{16}x^{-5/4} + \frac{3}{4}x^{-7/4}$
$= \dfrac{\frac{-3}{16}\sqrt{x} + \frac{3}{4}}{x^{7/4}}$
The critical number here is $x = 16$. We find that $f''(x) > 0$ on the interval $(0, 16)$ (so $f(x)$ is concave up on this interval) and $f''(x) < 0$ on the interval $(16, \infty)$ (so $f(x)$ is concave down on this interval).



22. $f(x) = x^{2/3} - 4x^{1/3}$
$f'(x) = \frac{2}{3}x^{-1/3} - \frac{4}{3}x^{-2/3}$
$= \dfrac{\frac{2}{3}x^{1/3} - \frac{4}{3}}{x^{2/3}}$
So $x = 0$ and $x = 8$ are critical numbers.
$f'(-1) < 0$ so $f(x)$ is decreasing for $x < 0$.
$f'(1) < 0$ so $f(x)$ is decreasing for $0 < x < 8$.
$f'(27) > 0$ so $f(x)$ is increasing on $8 < x$.
$f''(x) = -\frac{2}{9}x^{-4/3} + \frac{8}{9}x^{-5/3}$
$= \dfrac{-\frac{2}{9}x^{1/3} + \frac{8}{9}}{x^{5/3}}$

The critical numbers here are $x = 0$ and $x = 64$.
$f''(-1) < 0$ so $f(x)$ is concave down on $(-\infty, 0)$.
$f''(1) > 0$ so $f(x)$ is concave up on $(0, 64)$.
$f''(125) < 0$ so $f(x)$ is concave down on $(64, \infty)$.



23. The easiest way to sketch this graph is to notice that

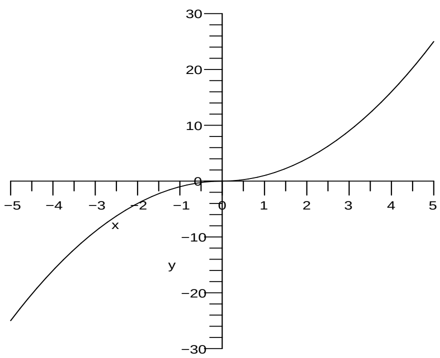
$$f(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

Since

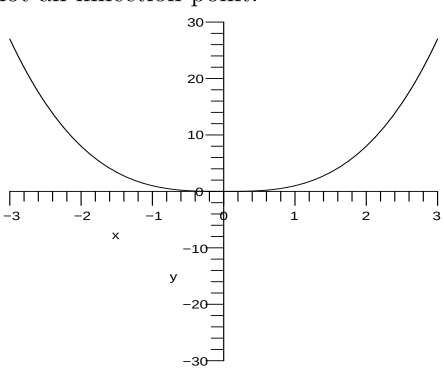
$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

there is a critical point at $x = 0$. However, it is neither a local maximum nor a local minimum.

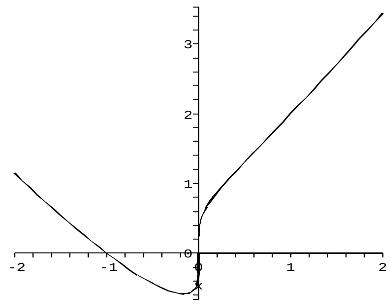
Since
$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}$$
there is an inflection point at the origin. Note that the second derivative does not exist at $x = 0$.



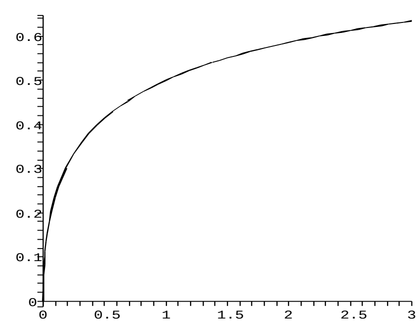
24. The easiest way to sketch this graph is to notice that
$$f(x) = x^2|x| = \begin{cases} -x^3 & x < 0 \\ x^3 & x \ge 0 \end{cases}$$
since
$$f'(x) = \begin{cases} -3x^2 & x < 0 \\ 3x^2 & x \ge 0 \end{cases}$$
there is a critical point (and local minimum) at $x = 0$. Since
$$f''(x) = \begin{cases} -6x & x < 0 \\ 6x & x \ge 0 \end{cases}$$
there is a critical point at the origin but this is not an inflection point.



25. $f(x) = x^{1/5}(x+1) = x^{6/5} + x^{1/5}$
$f'(x) = \frac{6}{5}x^{1/5} + \frac{1}{5}x^{-4/5}$
$\quad = \frac{1}{5}x^{-4/5}(6x+1)$
$f''(x) = \frac{6}{25}x^{-4/5} - \frac{4}{25}x^{-9/5}$
$\quad = \frac{2}{25}x^{-9/5}(3x-2)$
Note that $f(0) = 0$, and yet the derivatives do not exist at $x = 0$. This means that there is a vertical tangent line at $x = 0$. The first derivative is negative for $x < -1/6$ and positive for $-1/6 < x < 0$ and $x > 0$. The second derivative is positive for $x < 0$ and $x > 2/3$, and negative for $0 < x < 2/3$. Thus, there is a local minimum at $x = -1/6$ and inflection points at $x = 0$ and $x = 2/3$.

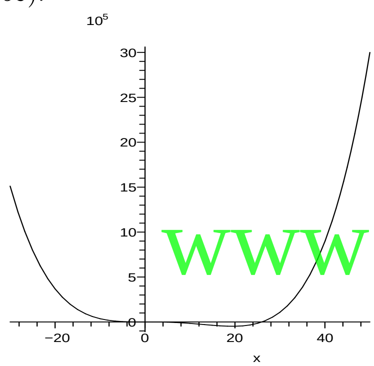


26. $f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x}}$
The domain of $f(x)$ is $\{x | x \ge 0\}$.
$$f'(x) = \frac{\frac{1}{2}x^{-1/2}(1+\sqrt{x}) - \sqrt{x}(\frac{1}{2}x^{-1/2})}{(x+\sqrt{x})^2}$$
$$= \frac{x^{-1/2}}{2(1+\sqrt{x})^2}$$
The only critical point is $x = 0$, which we need not consider because of the domain. Since $f'(1) > 0$, $f(x)$ is increasing on $(0, \infty)$.
$$f''(x) = \frac{-x^{-3/2}(1+\sqrt{x})^2 - 2x^{-1/2}(1+\sqrt{x})x^{-1/2}}{4(1+\sqrt{x})^4}$$
$$= \frac{-(x^{-1/2}+3)}{4x(1+\sqrt{x})^3}$$
The critical numbers are $x = 0$ (which we again ignore) and $x = 1/9$. Since $f''(1) < 0$ and $f''(1/16) < 0$, $f(x)$ is concave down on $(0, \infty)$.



27. $f(x) = x^4 - 26x^3 + x$
$f'(x) = 4x^3 - 78x^2 + 1$
The critical numbers are approximately -0.1129, 0.1136 and 19.4993.
$f'(-1) < 0$ implies $f(x)$ is decreasing on

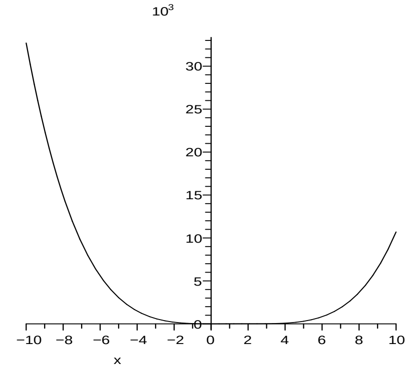
$(-\infty, -0.1129)$.
$f'(0) > 0$ implies $f(x)$ is increasing on $(-0.1129, 0.1136)$.
$f'(1) < 0$ implies $f(x)$ is decreasing on $(0.1136, 19.4993)$.
$f'(20) > 0$ implies $f(x)$ is increasing on $(19.4993, \infty)$.
Thus $f(x)$ has local minimums at $x = -0.1129$ and $x = 19.4993$ and a local maximum at $x = 0.1136$.
$f''(x) = 12x^2 - 156x = x(12x - 156)$
The critical numbers are $x = 0$ and $x = 13$.
$f''(-1) > 0$ implies $f(x)$ is concave up on $(-\infty, 0)$.
$f''(1) < 0$ implies $f(x)$ is concave down on $(0, 13)$.
$f''(20) > 0$ implies $f(x)$ is concave up on $(13, \infty)$.



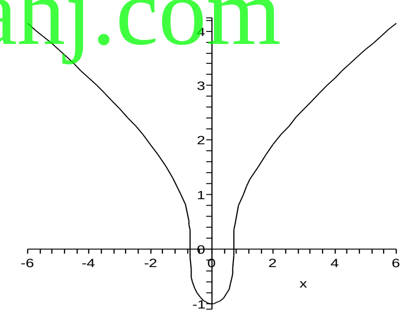
28. $f(x) = 2x^4 - 11x^3 + 17x^2$
$f'(x) = 8x^3 - 33x^2 + 34x$
$\quad = x(8x - 17)(x - 2)$
The critical numbers are $x = 0$, $x = 2$ and $x = 17/8$.
$f''(x) = 24x^2 - 66x + 34$
$f''(0) > 0$ implies $f(x)$ is concave up at $x = 0$ so $f(x)$ has a local minimum here and $f(x)$ is decreasing on $(-\infty, 0)$.
$f''(2) < 0$ implies $f(x)$ is concave down at $x = 2$ so $f(x)$ has a local maximum here and $f(x)$ is increasing on $(0, 2)$.
$f''(17/8) > 0$ implies $f(x)$ is concave up at $x = 17/8$ so $f(x)$ has a local minimum here and $f(x)$ is decreasing on $(2, 17/8)$ and increasing on $(17/8, \infty)$.
$f''(x) = 2(12x^2 - 33x + 17)$
The critical numbers are

$$x = \frac{33 \pm \sqrt{273}}{24} = 2.0635,\ 0.6866.$$

So $f(x)$ is concave up on $(-\infty, 0.6866)$ and $(2.0635, \infty)$ and $f(x)$ is concave down on $(0.6866, 2.0635)$.



29. $y = \sqrt[3]{2x^2 - 1}$
$y' = \dfrac{4x}{3(2x^2 - 1)^{2/3}} = 0$ at $x = 0$ and is undefined at $x = \pm\sqrt{1/2}$.
$y'' = \dfrac{-4(2x^2 + 3)}{9(2x^2 - 1)^{5/3}}$ is never 0, and is undefined where y' is.
The function changes concavity at $x = \pm\sqrt{1/2}$, so these are inflection points. The slope does not change at these values, so they are not extrema. The Second Derivative Test shows that $x = 0$ is a minimum.

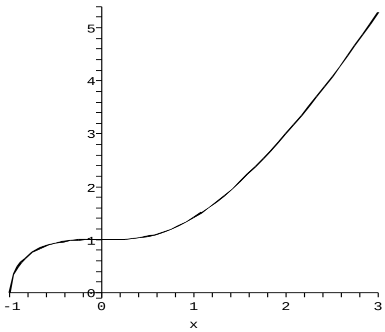


30. $f(x) = \sqrt{x^3 + 1}$
$f(x)$ is defined for $x \ge -1$.
$f'(x) = \frac{1}{2}(x^3 + 1)^{-1/2}(3x^2)$
The critical numbers are $x = -1$ (which we ignore because of the domain) and $x = 0$.
$f'(-1/2) > 0$ so $f(x)$ is increasing on $(-1, 0)$.
$f'(1) > 0$ so $f(x)$ is also increasing on $(0, \infty)$ so $f(x)$ has no relative extrema.
$f''(x) =$

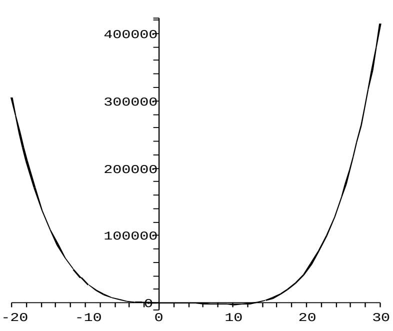
$$\frac{3}{2} \cdot \frac{2x(x^3 + 1)^{1/2} - x^2\frac{1}{2}(x^3 + 1)^{-1/2}3x^2}{x^3 + 1}$$
$$= \frac{2x(x^3 + 1) - \frac{3}{2}x^4}{(x^3 + 1)^{3/2}}$$
$$= \frac{-\frac{1}{2}x^4 + 2x}{(x^3 + 1)^{3/2}}$$

The critical numbers are $x = 0$ and $x = 4^{1/3}$ (and $x = -1$, which we need not consider).

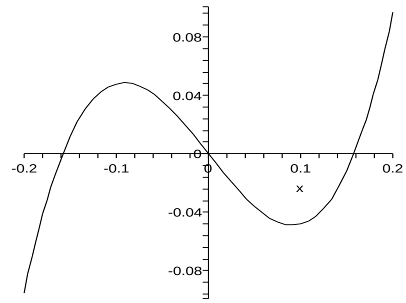
$f''(-1/2) < 0$ so $f(x)$ is concave down on $(-1, 0)$. $f''(1) > 0$ so $f(x)$ is concave up on $(0, 4^{1/3})$. $f''(2) > 0$ so $f(x)$ is concave up on $(4^{1/3}, \infty)$.



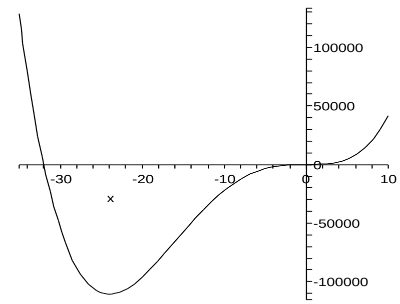
31. $f(x) = x^4 - 16x^3 + 42x^2 - 39.6x + 14$
$f'(x) = 4x^3 - 48x^2 + 84x - 39.6$
$f''(x) = 12x^2 - 96x + 84$
$= 12(x^2 - 8x + 7)$
$= 12(x - 7)(x - 1)$
$f'(x) > 0$ on $(.8952, 1.106) \cup (9.9987, \infty)$
$f'(x) < 0$ on $(-\infty, .8952) \cup (1.106, 9.9987)$
$f''(x) > 0$ on $(-\infty, 1) \cup (7, \infty)$
$f''(x) < 0$ on $(1, 7)$
f is increasing on $(.8952, 1.106)$ and on $(9.9987, \infty)$, decreasing on $(-\infty, .8952)$ and on $(1.106, 9.9987)$, concave up on $(-\infty, 1) \cup (7, \infty)$, concave down on $(1, 7)$, $x = .8952$, 9.9987 are local min, $x = 1.106$ is local max, $x = 1, 7$ are inflection points.



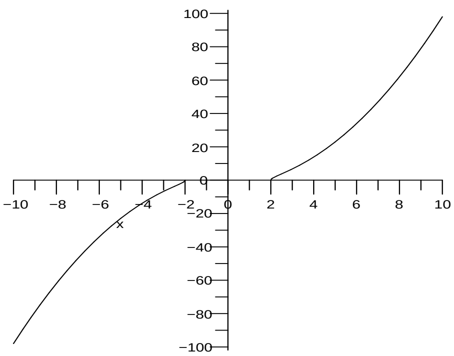
32. $y = x^4 + 32x^3 - 0.02x^2 - 0.8x$
$y' = 4x^3 + 96x^2 - 0.04x - 0.8 = 0$ at approximately $x = -24$, -0.09125, and 0.09132.
$y'' = 12x^2 + 192x - 0.04 = 0$ at approximately $x = 16.0002$ and 0.0002, and changes sign at these values, so these are inflection points. The Second Derivative Test shows that $x = -24$ and 0.09132 are minima, and that $x = -0.09125$ is a maxima. The extrema near $x = 0$ look like this:



The global behavior looks like this:



33. $f(x) = x\sqrt{x^2 - 4}$; f undefined on $(-2, 2)$
$$f'(x) = \sqrt{x^2 - 4} + x\left(\tfrac{1}{2}\right)(x^2 - 4)^{-1/2}(2x)$$
$$= \sqrt{x^2 - 4} + \frac{x^2}{\sqrt{x^2 - 4}}$$
$$= \frac{2x^2 - 4}{\sqrt{x^2 - 4}}$$
$$f''(x) = \frac{4x\sqrt{x^2 - 4} - (2x^2 - 4)\frac{1}{2}(x^2 - 4)^{-1/2}(2x)}{x^2 - 4}$$
$$= \frac{4x(x^2 - 4) - (2x^2 - 4)x}{(x^2 - 4)^{3/2}}$$
$$= \frac{2x^3 - 12x}{(x^2 - 4)^{3/2}} = \frac{2x(x^2 - 6)}{(x^2 - 4)^{3/2}}$$
$f'(x) > 0$ on $(-\infty, -2) \cup (2, \infty)$
$f''(x) > 0$ on $\left(-\sqrt{6}, 2\right) \cup \left(\sqrt{6}, \infty\right)$
$f''(x) < 0$ on $\left(-\infty, -\sqrt{6}\right) \cup \left(2, \sqrt{6}\right)$
f is increasing on $(-\infty, -2)$ and on $(2, \infty)$, concave up on $\left(-\sqrt{6}, -2\right) \cup \left(\sqrt{6}, \infty\right)$, concave down on $\left(-\infty, -\sqrt{6}\right) \cup \left(2, \sqrt{6}\right)$, $x = \pm\sqrt{6}$ are inflection points.



34. $f(x) = \dfrac{2x}{\sqrt{x^2+4}}$

$f'(x) = \dfrac{2\sqrt{x^2+4} - 2x(\frac{1}{2})(x^2+4)^{-1/2}2x}{(x^2+4)}$

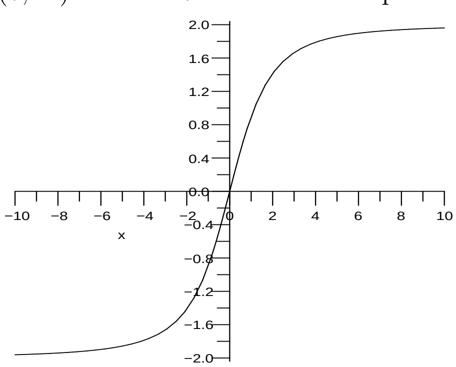
$= \dfrac{8}{(x^2+4)^{3/2}}$

$f'(x)$ is always positive, so there are no critical points and $f(x)$ is always increasing.

$f''(x) = 8(-\frac{3}{2})(x^2+4)^{-5/2}(2x)$

$= \dfrac{-24x}{(x^2+4)^{5/2}}$

The only critical point is $x = 0$. Since $f''(-1) > 0$, $f(x)$ is concave up on $(-\infty, 0)$. Also $f''(1) < 0$, so $f(x)$ is concave down on $(0, \infty)$ and $x = 0$ is an inflection point for f.



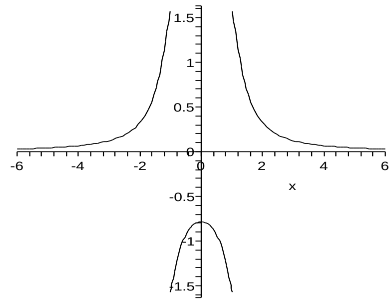
35. The function has horizontal asymptote $y = 0$, and is undefined at $x = \pm 1$.

$y' = \dfrac{-2x}{x^4 - 2x^2 + 2} = 0$

only when $x = 0$.

$y'' = \dfrac{2(3x^4 - 2x^2 - 2)}{(x^4 - 2x^2 + 2)^2} = 0$

at approximately $x = \pm 1.1024$ and changes sign there, so these are inflection points (very easy to miss by looking at the graph). The Second Derivative Test shows that $x = 0$ is a local maximum.



36. $f(x) = e^{-2x}\cos x$

$f'(x) = -2e^{-2x}\cos x - e^{-2x}\sin x$

$= e^{-x}(-2\cos x - \sin x)$

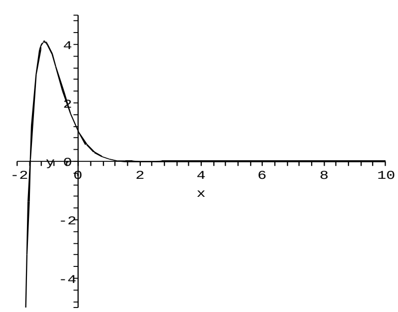
$f''(x) = -2e^{-2x}(-2\cos x - \sin x)$

$+ e^{-2x}(2\sin x - \cos x)$

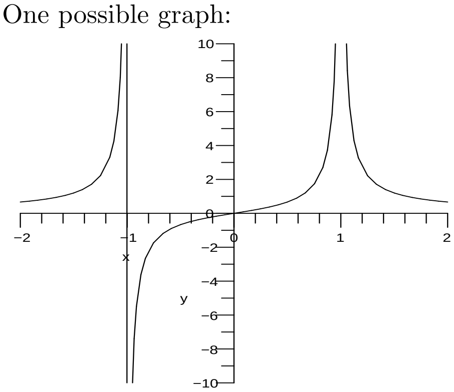
$= e^{-2x}(4\sin x + 3\cos x)$

$f'(x) = 0$ when $\sin x = -2\cos x$ so when $x = k\pi + \tan^{-1}(-2)$ for any integer k.

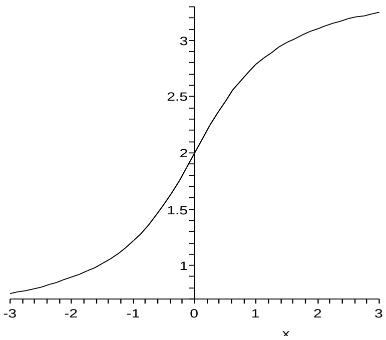
$f''(2k\pi + \tan^{-1}(-2)) < 0$ so there are local maxima at all such points, while $f''((2k+1)\pi + \tan^{-1}(-2)) > 0$, so there are local minima at all such points. $f''(x) = 0$ when $4\sin x = -3\cos x$ or $x = k\pi + \tan^{-1}(-3/4)$ for any integer k. All such points x are inflection points.



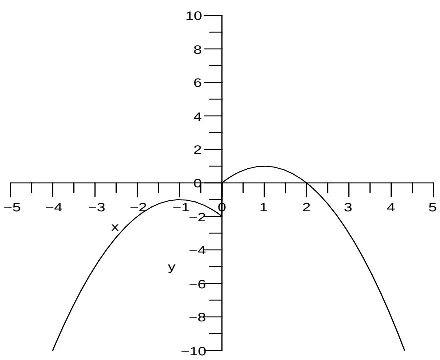
37. One possible graph:



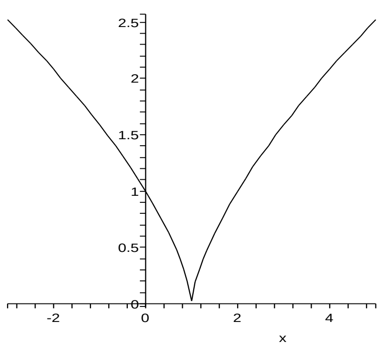
38. One possible graph:



39. One possible graph:



40. One possible graph:



41. $f(x) = ax^3 + bx^2 + cx + d$
$f'(x) = 3ax^2 + 2bx + c$
$f''(x) = 6ax + 2b$
Thus, $f''(x) = 0$ for $x = -b/3a$. Since f'' changes sign at this point, f has an inflection point at $x = -b/3a$. Note that $a \neq 0$.

For the quartic function (where again $a \neq 0$),
$f(x) = ax^4 + bx^3 + cx^2 + dx + e$
$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$
$$\begin{aligned} f''(x) &= 12ax^2 + 6bx + 2c \\ &= 2(6ax^2 + 3bx + c) \end{aligned}$$
The second derivative is zero when
$$\begin{aligned} x &= \frac{-3b \pm \sqrt{9b^2 - 24ac}}{12a} \\ &= \frac{-3b \pm \sqrt{3(3b^2 - 8ac)}}{12a} \end{aligned}$$
There are two distinct solutions to the previous equation (and therefore two inflection points) if and only if $3b^2 - 8ac > 0$.

42. Since $f'(0) = 0$ and $f''(0) > 0$, $f(x)$ must have a local minimum at $x = 0$. Since we also know that $f(0) = 0$, this means that there is some neighborhood (possibly very small) of 0 such that for all x in this neighborhood (exluding $x = 0$), $f(x) > 0$.

Similarly, $g'(0) = 0$ and $g''(0) < 0$ implies that $g(x)$ must have a local maximum at $x = 0$. Again we know that $g(0) = 0$, so there is some neighborhood of 0 such that for all x in this neighborhood (exluding $x = 0$), $g(x) < 0$.

On the smaller of these two neighborhoods, we know that $g(x) < 0 < f(x)$.

43. Let $f(x) = -1 - x^2$. Then
$f'(x) = -2x$
$f''(x) = -2$
so f is concave down for all x, but
$-1 - x^2 = 0$ has no solution.

44. The statement is true.

45. $f(x)$ is concave up on $(-\infty, -0.5)$ and $(0.5, \infty)$; $f(x)$ is concave down on $(-0.5, 0.5)$. $f(x)$ is decreasing on the intervals $(-\infty, 1)$ and $(0, 1)$; increasing on the intervals $(-1, 0)$ and $(1, \infty)$. $f(x)$ has local maxima at 0 and minima at -1 and 1. Inflection points of $f(x)$ are -0.5 and 0.5.

46. $f(x)$ is concave up on $(1, \infty)$; $f(x)$ is concave down on $(-\infty, 1)$. $f(x)$ is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$; decreasing on the intervals $(0, 2)$. Inflection point of $f(x)$ is 1.

47. (a) **For #45 :**
The interval of increase is $(-\infty, -1.5)$ and $(1.5, \infty)$. The interval of decrease is$(-1.5, 1.5)$. Minima at $x = 1.5$ and Maxima at $x = -1.5$. It is concave up for $(-1, 0) \cup (1, \infty)$. It is concave down for $(-\infty, -1) \cup (0, 1)$. The points of inflection are $x = 0$ and ± 1.

For #46:
The interval of increase is $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup (3, \infty)$. The interval of decrease is $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 3\right)$ Minima at $x = \frac{1}{2}$and Maxima at $x = -\frac{1}{2}, 3$. It is concave up for$(-\infty, 0) \cup (2, \infty)$. It is concave down for$(0, 2)$. The points of inflection

are $x = 0$, 2..

(b) **For #45:**
It is concave up for $(-\infty, -1.5) \cup (1.5, \infty)$. It is concave down for $(-1.5, 1.5)$. The points of inflection are $x = 0$ and ± 1.5.

For #46:
It is concave up for $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup (3, \infty)$. It is concave down for $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 3\right)$. The points of inflection are $x = \pm\frac{1}{2}$, 3..

48. If $f''(c) < 0$, then f' is decreasing at c. Because $f'(c) = 0$, this means that $f' > 0$ to the left of c and $f' < 0$ to the right of c. Therefore, by the First Derivative Test, $f(c)$ is a local maximum. The proof of the second claim is similar.

49. Add and subtract 16 to complete square.
$x^4 - 8x^2 + 10$
$= x^4 - 8x^2 + 16 + 10 - 16$
$= \left(x^2 - 4\right)^2 - 6$
Therefore, absolute minimum occurs when $\left(x^2 - 4\right)^2 = 0$. That is absolute minimum is -6 and occurs when $x = \pm 2$.

Similarly, add and subtract 9 to $x^4 - 6x^2 + 1$.
$x^4 - 6x^2 + 1$
$= x^4 - 6x^2 + 9 + 1 - 9$
$= \left(x^2 - 3\right)^2 - 8$

Therefore, absolute minimum occurs when $\left(x^2 - 3\right)^2 = 0$. That is absolute minimum is -8 and occurs when$x = \pm\sqrt{3}$.

50. $f(x) = x^4 + bx^3 + cx^2 + dx + 2$
$f'(x) = 4x^3 + 3bx^2 + 2cx + d$
$f''(x) = 12x^2 + 6bx + 2c$

To find inflection points, solve $f''(x) = 0$.
$x = \dfrac{-6b \pm \sqrt{36b^2 - 96c}}{24}$
x is real only if $36b^2 - 96c > 0$
$\Rightarrow c < \dfrac{3}{8}b^2$
The critical numbers are
$x = \dfrac{-6b + \sqrt{36b^2 - 96c}}{24}$
and
$x = \dfrac{-6b - \sqrt{36b^2 - 96c}}{24}$
Therefore sum of x-coordinates
$= \dfrac{-6b + \sqrt{36b^2 - 96c}}{24} + \dfrac{-6b - \sqrt{36b^2 - 96c}}{24}$
$= \dfrac{-6b + \sqrt{36b^2 - 96c} - 6b - \sqrt{36b^2 - 96c}}{24}$
$= \dfrac{-12b}{24} = -\dfrac{b}{2}$

51. We need to know $w'(0)$ to know if the depth is increasing.

52. We assume the sick person's temperature is too high, and not too low. We do need to know $T'(0)$ in order to tell which is better.

If $T''(0) = 2$ and $T' > 0$, the person's temperature is rising alarmingly.

If $T''(0) = -2$ and $T' > 0$, the person's temperature is increasing, but leveling off.
Negative T'' is better if $T' > 0$.

If $T''(0) = 2$ and $T' < 0$, the person's temperature is decreasing and leveling off.

If $T''(0) = -2$ and $T' < 0$, the person's temperature is dropping too steeply to be safe.
Positive T'' is probably better if $T' < 0$.

53. $s(x) = -3x^3 + 270x^2 - 3600x + 18000$
$s'(x) = -9x^2 + 540x - 3600$
$s''(x) = -18x + 540 = 0$
$x = 30$. This is a max because the graph of $s'(x)$ is a parabola opening down. So spend \$30,000 on advertising to maximize the rate of change of sales. This is also the inflection point of $s(x)$.

54. $Q'(t)$ measures the number of units produced per hour. If this number is larger, the worker is more efficient.
$Q'(t) = -3t^2 + 12t + 12$ will be maximized where
$Q'' = -6t + 12 = 0$, or $t = 2$ hours. (This is a maximum by the First Derivative Test.) It is reasonable to call this inflection point the point of diminishing returns, because after this point, the efficiency of the worker decreases.

55. $C(x) = .01x^2 + 40x + 3600$
$\overline{C}(x) = \dfrac{C(x)}{x} = .01x + 40 + 3600x^{-1}$
$\overline{C}'(x) = .01 - 3600x^{-2} = 0$
$x = 600$. This is a min because $\bar{C}''(x) = 7200x^{-3} > 0$ for $x > 0$, so the graph is concave up. So manufacture 600 units to minimize average cost.

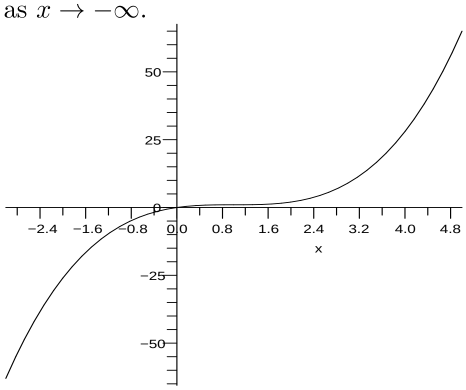
56. Solving $c' = 0$ yields $t = 19.8616$. The Second Derivative Test shows this is a maximum.

Solving $c'' = 0$ yields $t = 41.8362$. Suppose a second drug produced a similar plasma concentration graph, with the same maximum, but a later inflection point. Then the plasma concentration decays faster for the second drug, since it takes longer for the rate of decay to level off.

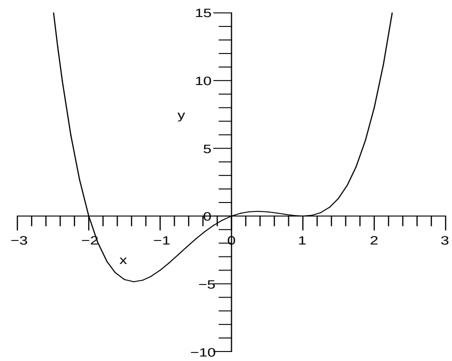
57. Since the tangent line points above the sun, the sun appears higher in the sky than it really is.

3.6 Overview of Curve Sketching

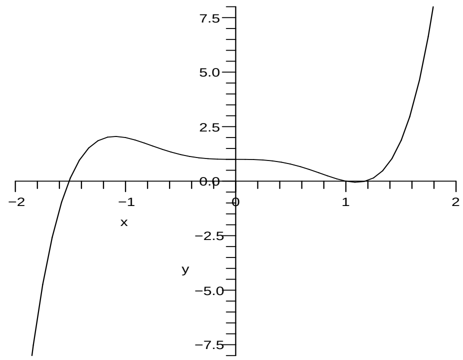
1. $f(x) = x^3 - 3x^2 + 3x$
$= x(x^2 - 3x + 3)$
The only x-intercept is $x = 0$; the y-intercept is $(0, 0)$.
$f'(x) = 3x^2 - 6x + 3$
$= 3(x^2 - 2x + 1) = 3(x - 1)^2$
$f'(x) > 0$ for all x, so $f(x)$ is increasing for all x and has no local extrema.
$f''(x) = 6x - 6 = 6(x - 1)$
There is an inflection point at $x = 1$: $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.
Finally, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.



2. $f(x) = x^4 - 3x^2 + 2x$
$= x(x^3 - 3x + 2)$
The x-intercepts are $x = -2$, $x = 1$ and $x = 0$; the y-intercept is (0, 0).
$f'(x) = 4x^3 - 6x + 2$
$= 2(2x^3 - 3x + 1)$
The critical numbers are $x = -1.366$, 0.366 and 1.
$f'(x) > 0$ on $(-1.366, 0.366)$ and $(1, \infty)$, so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(-\infty, -1.366)$ and $(0.366, 1)$, so $f(x)$ is decreasing on these intervals. Thus $f(x)$ has local minima at $x = -1.366$ and $x = 1$ and a local maximum at $x = 0.366$.
$f''(x) = 12x^2 - 6 = 6(2x^2 - 1)$
The critical numbers here are $x = \pm 1/\sqrt{2}$. $f''(x) > 0$ on $(-\infty, -1/\sqrt{2})$ and $(1/\sqrt{2}, \infty)$ so $f(x)$ is concave up on these intervals. $f''(x) < 0$ on $(-1/\sqrt{2}, 1/\sqrt{2})$ so $f(x)$ is concave down on this interval. Thus $f(x)$ has inflection points at $x = \pm 1/\sqrt{2}$.
Finally, $f(x) \to \infty$ as $x \to \pm\infty$.



3. $f(x) = x^5 - 2x^3 + 1$
The x-intercepts are $x = 1$ and $x \approx -1.5129$; the y-intercept is $(0, 1)$.
$f'(x) = 5x^4 - 6x^2 = x^2(5x^2 - 6)$
The critical numbers are $x = 0$ and $x = \pm\sqrt{6/5}$. Plugging values from each of the intervals into $f'(x)$, we find that $f'(x) > 0$ on $(-\infty, -\sqrt{6/5})$ and $(\sqrt{6/5}, \infty)$ so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(-\sqrt{6/5}, 0)$ and $(0, \sqrt{6/5})$ so $f(x)$ is decreasing on these intervals. Thus $f(x)$ has a local maximum at $-\sqrt{6/5}$ and a local minimum at $\sqrt{6/5}$.
$f''(x) = 20x^3 - 12x = 4x(5x^2 - 3)$
The critical numbers are $x = 0$ and $x = \pm\sqrt{3/5}$. Plugging values from each of the intervals into $f''(x)$, we find that $f''(x) > 0$ on $(-\sqrt{3/5}, 0)$ and $(\sqrt{3/5}, \infty)$ so $f(x)$ is concave up on these intervals. $f''(x) < 0$ on $(-\infty, -\sqrt{3/5})$ and $(0, \sqrt{3/5})$ so $f(x)$ is concave down on these intervals. Thus $f(x)$ has inflection points at all three of these critical numbers.
Finally, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.



4. $f(x) = x^4 + 4x^3 - 1$

The x-intercepts are $x \approx -4.01541$ and $x \approx 0.6012$; the y-intercept is $(0, -1)$.

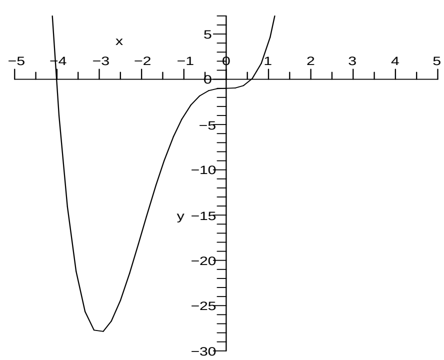
$f'(x) = 4x^3 + 12x^2 = 4x^2(x + 3)$

The critical numbers are $x = 0$ and $x = -3$. Plugging values from each of the intervals into $f'(x)$, we find that $f'(x) > 0$ on $(-3, 0)$ and $(0, \infty)$ so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(-\infty, -3)$ so $f(x)$ is decreasing on these intervals. Thus $f(x)$ has a local minimum at -3.

$f''(x) = 12x^2 + 24x = 12x(x + 2)$

The critical numbers are $x = 0$ and $x = -2$. Plugging values from each of the intervals into $f''(x)$, we find that $f''(x) > 0$ on $(-\infty, -2)$ and $(0, \infty)$ so $f(x)$ is concave up on $(-\infty, -2)$ and $(0, \infty)$. $f''(x) < 0$ on $(-2, 0)$ so $f(x)$ is concave down on $(-2, 0)$. The graph has inflection points at -2 and 0.

Finally, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.



5. $f(x) = x + \dfrac{4}{x} = \dfrac{x^2 + 4}{x}$

This function has no x- or y-intercepts. The domain is $\{x | x \neq 0\}$.

$f(x)$ has a vertical asymptote at $x = 0$ such that $f(x) \to -\infty$ as $x \to 0^-$ and $f(x) \to \infty$ as $x \to 0^+$.

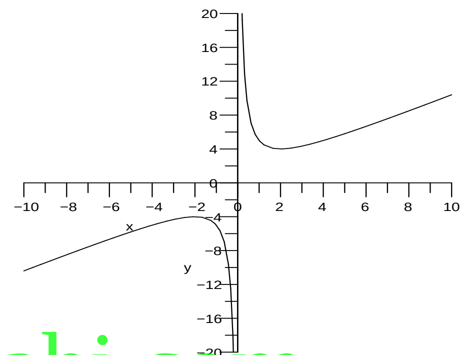
$f'(x) = 1 - 4x^{-2} = \dfrac{x^2 - 4}{x^2}$

The critical numbers are $x = \pm 2$. We find that $f'(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$ so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(-2, 0)$ and $(0, 2)$, so $f(x)$ is decreasing on these intervals. Thus $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 2$.

$f''(x) = 8x^{-3}$

$f''(x) < 0$ on $(-\infty, 0)$ so $f(x)$ is concave down on this interval and $f''(x) > 0$ on $(0, \infty)$ so $f(x)$ is concave up on this interval, but $f(x)$ has an asymptote (not an inflection point) at $x = 0$.

Finally, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



6. $f(x) = \dfrac{x^2 - 1}{x} = x - \dfrac{1}{x}$

There are x-intercepts at $x = \pm 1$, but no y-intercepts. The domain is $\{x | x \neq 0\}$.

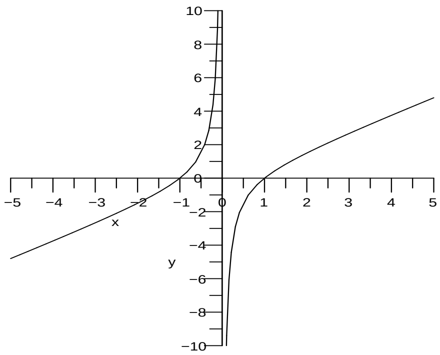
$f(x)$ has a vertical asymptote at $x = 0$ such that $f(x) \to \infty$ as $x \to 0^-$ and $f(x) \to -\infty$ as $x \to 0^+$.

$f'(x) = 1 + x^{-2} > 0$, So there is no critical numbers. $f(x)$ is increasing function.

$f''(x) = -2x^{-3}$

$f''(x) > 0$ on $(-\infty, 0)$ so $f(x)$ is concave up on this interval and $f''(x) < 0$ on $(0, \infty)$ so $f(x)$ is concave down on this interval, but $f(x)$ has an vertical asymptote (not an inflection point) at $x = 0$.

Finally, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



7. $f(x) = \dfrac{x^2+4}{x^3}$ has no x-interscept and no y-interscept. The domain of f includes all real numbers $x \neq 0$. $f(x)$ has a vertical asymptote at $x = 0$

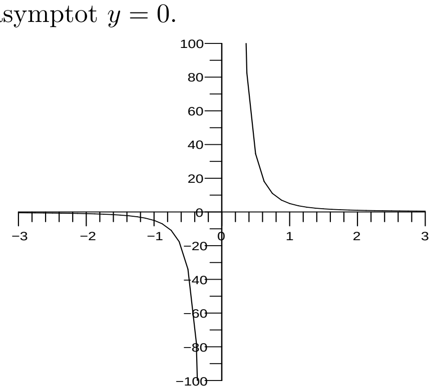
$$f'(x) = \frac{2x\left(x^3\right) - \left(x^2+4\right)\left(3x^2\right)}{\left(x^3\right)^2}$$
$$= \frac{-\left(x^2+12\right)}{x^4}$$

Since $f'(x) = 0$ has no real roots, the graph has no extrema. $f'(x) < 0$ on $(-\infty, 0)$ and $(0, \infty)$ so $f(x)$ is decreasing on these intervals. $f''(x) = -\left[\dfrac{x^4(2x) - \left(x^2+12\right)\left(4x^3\right)}{\left(x^4\right)^2}\right]$

$$= \frac{2\left[x^2+24\right]}{x^5}$$

$f''(x) < 0$ on $(-\infty, 0)$ so $f(x)$ is concave down on this interval and $f''(x) > 0$ on $(0, \infty)$ so $f(x)$ is concave up on this interval, but $f(x)$ has an asymptote (not an inflection point) at $x = 0$.

Finally, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$. Therefore, the graph has horizontal asymptot $y = 0$.



8. $f(x) = \dfrac{x-4}{x^3}$

The graph has x-intercepts at $x = 4$, but no y-intercepts. The domain of f includes all real numbers $x \neq 0$. $f(x)$ has a vertical asymptote at $x = 0$

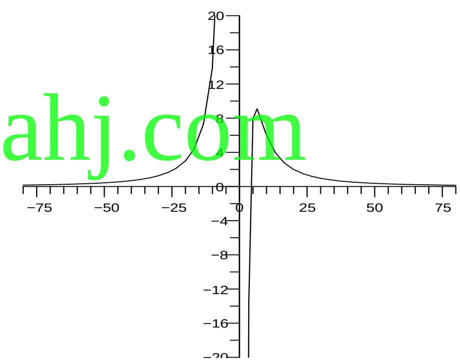
$$f'(x) = \frac{x^3 - (x-4)\left(3x^2\right)}{\left(x^3\right)^2}$$
$$= \frac{-2x+12}{x^4}$$

The critical numbers is $x = 6$. We find that $f'(x) > 0$ on $(-\infty, 0)$ and (0,6) so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(6, \infty)$, so $f(x)$ is decreasing on these intervals. Therefore, the graph has a local maximum at $x = 6$.

$$f''(x) = \frac{\left(x^4\right)(-2) - (-2x+12)\left(4x^3\right)}{\left(x^4\right)^2}$$
$$= \frac{6x-48}{x^5}$$

$f''(x) > 0$ on $(-\infty, 0)$ and $(8, \infty)$ so $f(x)$ is concave up on this interval and $f''(x) < 0$ on $(0, 8)$ so $f(x)$ is concave down on this interval, but $f(x)$ has an inflection point at $x = 8$.

Finally, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$. Therefore, the graph has horizontal asymptote $y = 0$.



9. $f(x) = \dfrac{2x}{x^2-1}$

The graph has x-intercept and y-intercept at (0, 0). The domain of f includes all real numbers $x = \pm 1$. $f(x)$ has vertical asymptotes at $x = \pm 1$.

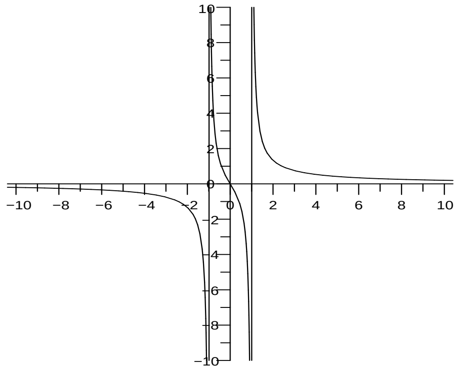
$$f'(x) = \frac{2\left(x^2-1\right) - (2x)(2x)}{\left(x^2-1\right)^2}$$
$$= \frac{-2\left(x^2+1\right)}{\left(x^2-1\right)^2}$$

Since $f'(x) = 0$ has no real roots, the graph has no extrema. $f'(x) < 0$ on $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ so $f(x)$ is decreasing on these intervals.

$$f''(x) = -2\left[\frac{2x\left(x^2-1\right)\left[x^2-1-2x^2-2\right]}{\left(x^2-1\right)^4}\right]$$
$$= \frac{4x\left[x^2+3\right]}{\left(x^2-1\right)^3}$$

$f''(x) > 0$ on $(-1, 0)$ and $(1, \infty)$ so $f(x)$ is concave up on this interval and $f''(x) < 0$ on

$(-\infty, -1)$ and $(0, 1)$ so $f(x)$ is concave down on this interval, but $f(x)$ has an inflection point at $x = 0$.
Finally, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$. Therefore, the graph has horizontal asymptote $y = 0$.



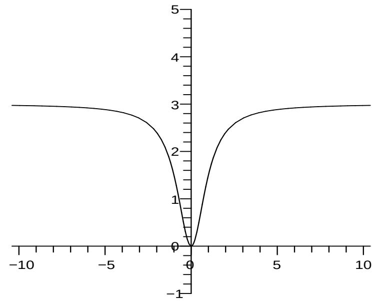
10. $f(x) = \dfrac{3x^2}{x^2+1}$
The graph has x-intercept and y-intercept at $(0,\,0)$. The domain of f includes all real numbers.

$$f'(x) = \frac{(x^2+1)(6x) - (3x^2)(2x)}{(x^2+1)^2} = \frac{6x}{(x^2+1)^2}$$

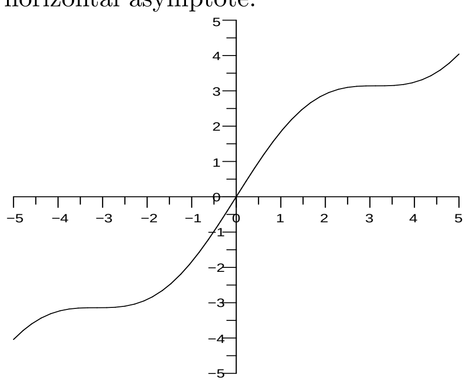
$f'(x) < 0$ on $(-\infty, 0)$ so $f(x)$ is decreasing on these intervals and $f'(x) > 0$ on $(0, \infty)$ so $f(x)$ is increasing on these interval.

$$f''(x) = \frac{(x^2+1)\left[6(x^2+1) - 24x^2\right]}{(x^2+1)^4} = \frac{6 - 18x^2}{(x^2+1)^3}$$

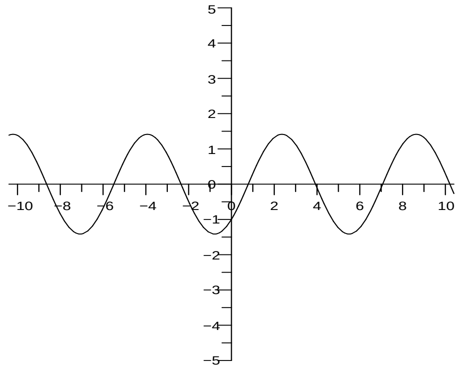
The critical numbers are $x = \pm\sqrt{\frac{1}{3}}$. We find that $f''(x) > 0$ on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$ so $f(x)$ is concave up on this interval and we find that $f''(x) < 0$ on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and $\left(\sqrt{\frac{1}{3}}, \infty\right)$ so $f(x)$ is concave down on this interval, but the graph has inflection points at $x = \pm\sqrt{\frac{1}{3}}$.
Finally, $f(x) \to 3$ as $x \to -\infty$ and $f(x) \to 3$ as $x \to \infty$. Therefore, the graph has horizontal asymptote at $y = 3$.



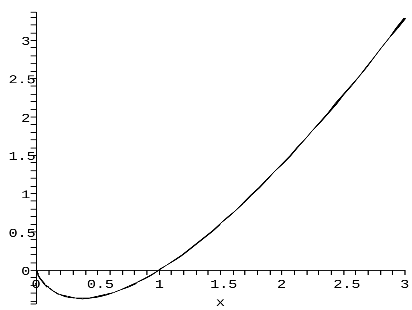
11. $f(x) = (x + \sin x)$
The graph has x-intercepts and y-intercepts at $(0,\,0)$. The domain of f includes all real numbers.
$f'(x) = 1 + \cos x \geq 0$,therefore the graph has no extrema and $f(x)$ is a increasing function.
$f''(x) = -\sin x$
$f''(x) < 0$ on $(2n\pi, (2n+1)\pi)$ so $f(x)$ is concave down on this interval and we find that $f''(x) > 0$ on $((2n+1)\pi, 2(n+1)\pi)$ so $f(x)$ is concave up on this interval, but the graph has inflection points at $x = n\pi$.
Finally, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. Therefore, the graph has no horizontal asymptote.



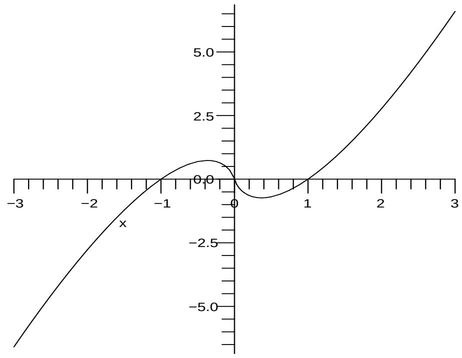
12. $f(x) = \sin x - \cos x$
$f'(x) = \cos x + \sin x$ is zero for $x = n\pi - \frac{\pi}{4}$.
$f''(x) = -\sin x + \cos x$
When n is even, $f''(x) > 0$ and so f is minimum at $x = n\pi - \frac{\pi}{4}$.
When n is odd, $f''(x) < 0$ and so f is maximum at $x = n\pi - \frac{\pi}{4}$.
$f''(x) = 0$ for $x = n\pi + \frac{\pi}{4}$. So inflection points are $n\pi + \frac{\pi}{4}$.
$f''(x) < 0$ on $\left(\frac{\pi}{4} + n\pi, \frac{5\pi}{4} + n\pi\right)$ so $f(x)$ is concave down on this interval and we find that $f''(x) > 0$ on $\left(\frac{5\pi}{4} + n\pi, \frac{9\pi}{4} + n\pi\right)$ so $f(x)$ is concave up on this interval.



13. $f(x) = x \ln x$
The domain is $\{x | x > 0\}$. There is an x-intercept at $x = 1$ and no y-intercept.
$f'(x) = \ln x + 1$
The only critical number is $x = e^{-1}$. $f'(x) < 0$ on $(0, e^{-1})$ and $f'(x) > 0$ on (e^{-1}, ∞) so $f(x)$ is decreasing on $(0, e^{-1})$ and increasing on (e^{-1}, ∞). Thus $f(x)$ has a local minimum at $x = e^{-1}$.
$f''(x) = 1/x$, which is positive for all x in the domain of f, so $f(x)$ is always concave up.
$f(x) \to \infty$ as $x \to \infty$.



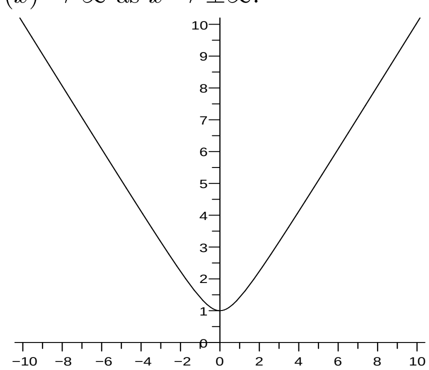
14. $f(x) = x \ln x^2$
The domain is $\{x | x \neq 0\}$. There are x-intercepts at $x = \pm 1$ but no y-intercept.
$f'(x) = \ln x^2 + 2$
The critical numbers are $x = \pm e^{-1}$. $f''(x) = 2/x$, so $x = -e^{-1}$ is a local maximum and $x = e^{-1}$ is a local minimum. $f(x)$ is increasing on $(-\infty, -e^{-1})$ and (e^{-1}, ∞); $f(x)$ is decreasing on $(-e^{-1}, 0)$ and $(0, e^{-1})$. $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.
$f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to \infty$.



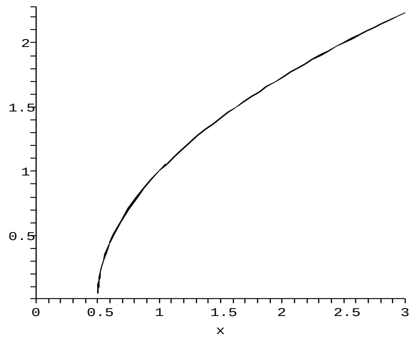
15. $f(x) = \sqrt{x^2 + 1}$
The y-intercept is $(0, 1)$. There are no x-intercepts.
$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2} 2x = \frac{x}{\sqrt{x^2+1}}$ The only critical number is $x = 0$. $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$ so $f(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. Thus $f(x)$ has a local minimum at $x = 0$.

$$f''(x) = \frac{\sqrt{x^2 + 1} - x\frac{1}{2}(x^2 + 1)^{-1/2} 2x}{x^2 + 1}$$
$$= \frac{1}{(x^2 + 1)^{3/2}}$$

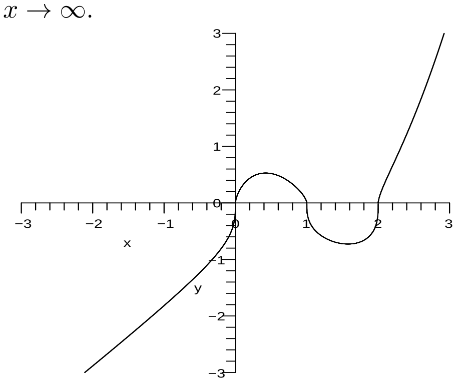
Since $f''(x) > 0$ for all x, we see that $f(x)$ is concave up for all x.
$f(x) \to \infty$ as $x \to \pm\infty$.



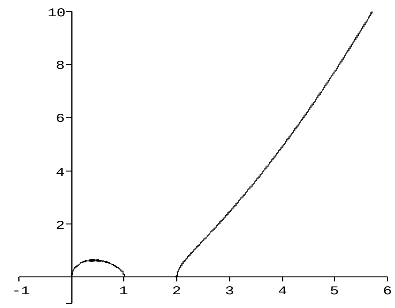
16. $f(x) = \sqrt{2x - 1}$
The domain is $\{x | x \geq 1/2\}$. There is an x-intercept at $x = 1/2$.
$f'(x) = \frac{1}{2}(2x - 1)^{-1/2} 2 = \frac{1}{\sqrt{2x-1}}$
$f'(x)$ is undefined at $x = 1/2$, but this is an endpoint of $f(x)$ and there are no other critical points. Since $f'(x)$ is positive for all x in the domain of f, we see that $f(x)$ is increasing for all x in the domain.
$f''(x) = -\frac{1}{2}(2x - 1)^{-3/2} 2 = \frac{-1}{(2x-1)^{3/2}}$
$f''(x) < 0$ for all x in the domain of f, so f is concave down for all x for which it is defined.
$f(x) \to \infty$ as $x \to \infty$.



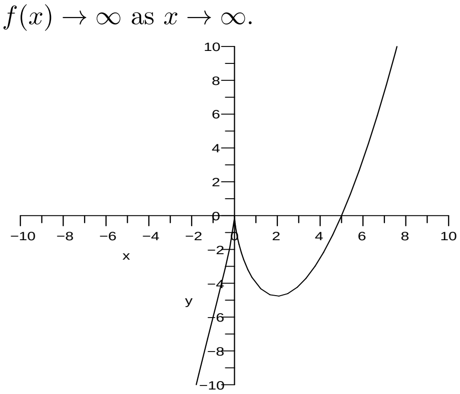
17. $f(x) = (x^3 - 3x^2 + 2x)^{1/3}$
$f'(x) = \dfrac{3x^2 - 6x + 2}{3(x^3 - 3x^2 + 2x)^{2/3}}$
There are critical numbers at $x = \dfrac{3 \pm \sqrt{3}}{3}$, 0, 1 and 2.
$f''(x) = \dfrac{-6x^2 + 12x - 8}{9(x^3 - 3x^2 + 2x)^{5/3}}$
with critical numbers $x = 0$, 1 and 2. $f''(x)$ changes sign at these values, so these are inflection points. The Second Derivative test shows that $x = \dfrac{3 + \sqrt{3}}{3}$ is a minimum, and $x = \dfrac{3 - \sqrt{3}}{3}$ is a maximum.
$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



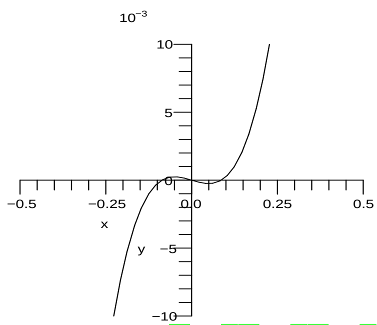
18. $f(x) = (x^3 - 3x^2 + 2x)^{1/2}$
$f(x)$ is defined for $0 \le x \le 1$ and $x \ge 2$. $f(x) \to \infty$ as $x \to \infty$.
$f'(x) = \dfrac{3x^2 - 6x + 2}{2(x^3 - 3x^2 + 2x)^{1/2}}$
There are critical numbers at $x = \dfrac{3 \pm \sqrt{3}}{3}$, 0, 1 and 2.
$f''(x) = \dfrac{3x^4 - 12x^3 + 12x^2 - 4}{4(x^3 - 3x^2 + 2x)^{3/2}}$
with critical numbers $x = 0$, 1 and 2 and $x \approx -0.4679$ and 2.4679. $f(x)$ is undefined at $x = -0.4679$, so we do not consider this point. $f''(x)$ changes sign at $x = 2.4679$, so this is an inflection point. The Second Derivative test shows that $x = \dfrac{3 - \sqrt{3}}{3}$ is a maximum.
At $x = 0$, 1, 2, $f(x)$ is minimum.



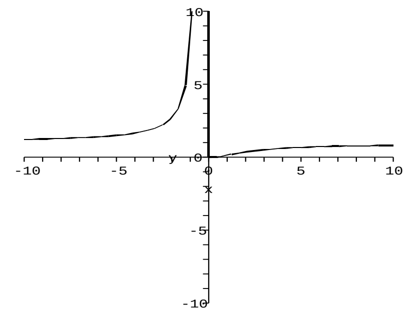
19. $f(x) = x^{5/3} - 5x^{2/3}$
The domain of f includes all real numbers.
$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = \frac{5}{3}\left(x^{\frac{2}{3}} - 2x^{-\frac{1}{3}}\right) = \frac{5}{3}\left(\frac{x-2}{x^{1/3}}\right)$$
Critical number is $x = 2$
$f'(x) > 0$ on $(-\infty, 0)$ and $(2, \infty)$. So $f(x)$ is increasing on these intervals.
$f'(x) < 0$ on $(0, 2)$ and so $f(x)$ is decreasing on this interval.
Therefore $f(x)$ is maximum at $x = 0$ and minimum at $x = 2$.
$$f''(x) = \frac{5}{3}\left(\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}x^{-\frac{4}{3}}\right) = \frac{10}{9}\left(x^{-\frac{1}{3}} + x^{-\frac{4}{3}}\right) = \frac{10}{9}\left(\frac{x+1}{x^{4/3}}\right)$$
The critical number is at $x = 0, -1$. $f''(x)$ changes sign at these values, so these are inflection points. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



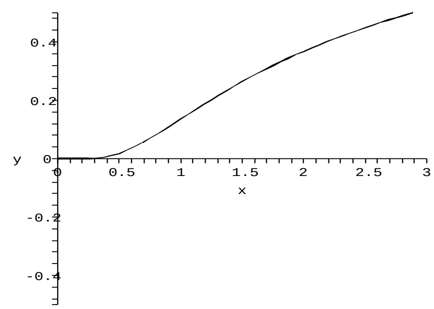
20. $f(x) = x^3 - \dfrac{3}{400}x = x(x^2 - \dfrac{3}{400})$
The y-intercept (also an x-intercept) is $(0,0)$ and there are also x-intercepts at $x = \pm\sqrt{3}/20$.
$f'(x) = 3x^2 - \dfrac{3}{400}$
The critical numbers are $x = \pm 1/20$.
$f''(x) = 6x$, so $x = -1/20$ is a local maximum and $x = 1/20$ is a local minimum. $f(x)$ is increasing on $(-\infty, -1/20)$ and $(1/20, \infty)$ and decreasing on $(-1/20, 1/20)$. It is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$, with an inflection point at $x = 0$.
$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



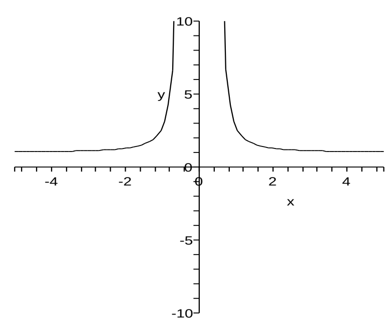
21. $f(x) = e^{-2/x}$
$$f'(x) = e^{-2/x}\left(\frac{2}{x^2}\right) = \frac{2}{x^2}e^{-2/x}$$
$$f''(x) = \frac{-4}{x^3}e^{-2/x} + \frac{2}{x^2}e^{-2/x}\left(\frac{2}{x^2}\right)$$
$$= \frac{4}{x^4}e^{-2/x} - \frac{4}{x^3}e^{-2/x}$$
$f'(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$
$f''(x) > 0$ on $(-\infty, 0) \cup (0, 1)$
$f''(x) < 0$ on $(1, \infty)$
f increasing on $(-\infty, 0)$ and on $(0, \infty)$, concave up on $(-\infty, 0) \cup (0, 1)$, concave down on $(1, \infty)$, inflection point at $x = 1$. f is undefined at $x = 0$.
$\lim\limits_{x\to 0^+} e^{-2/x} = \lim\limits_{x\to 0^+} \dfrac{1}{e^{2/x}} = 0$ and
$\lim\limits_{x\to 0^-} e^{-2/x} = \infty$
So f has a vertical asymptote at $x = 0$.
$\lim\limits_{x\to\infty} e^{-2/x} = \lim\limits_{x\to-\infty} e^{-2/x} = 1$
So f has a horizontal asymptote at $y = 1$.
Global graph of $f(x)$:



Local graph of $f(x)$:

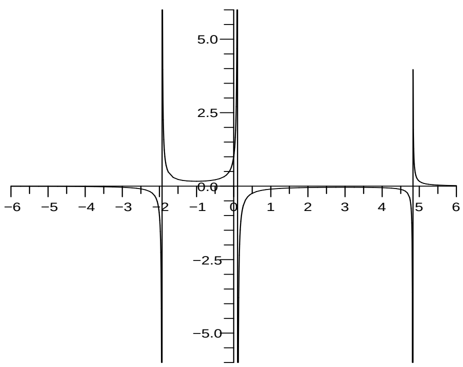


22. $f(x) = e^{1/x^2}$
The function has a vertical asymptote at $x = 0$ such that $f(x) \to \infty$ as x approaches 0 from the right or left. There is a horizontal asymptote of $y = 1$ as $x \to \pm\infty$.
$f'(x) = \dfrac{-2}{x^3} \cdot e^{1/x^2}$
$f'(x) > 0$ for $x < 0$, so $f(x)$ is increasing on $(-\infty, 0)$ and $f'(x) < 0$ for $x > 0$, so $f(x)$ is decreasing on $(-\infty, 0)$.
$f''(x) = \dfrac{2e^{1/x^2}(3x^2 + 2)}{x^6}$
is positive for all $x \neq 0$, so $f(x)$ is concave up for all $x \neq 0$.

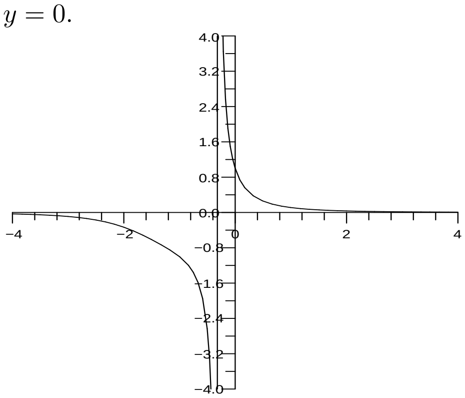


23. $f(x) = \dfrac{1}{x^3 - 3x^2 - 9x + 1}$
$f'(x) = -\dfrac{3x^2 - 6x - 9}{(x^3 - 3x^2 - 9x + 1)^2}$

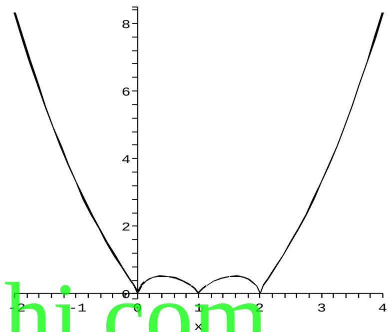
The critical numbers are $x = -1, 3$.
$$f''(x) = \frac{6\left(6x^4 - 4x^3 - 7x^2 + 12x + 2\right)}{\left(x^3 - 3x^2 - 9x + 1\right)^3}$$
The Second Derivative test shows that the graph has a local minimum at $x = -1$ and a local maximum at $x = 3$. The graph has a vertical asymptote at $x = -1.9304$. Similarly, the graph has vertical asymptotes at $x = 0.1074$ and 4.8231.
$f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$. Therefore, the graph has horizontal asymptote $y = 0$.



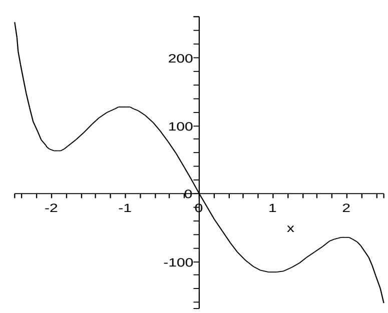
24. $f(x) = \dfrac{1}{x^3 + 3x^2 + 4x + 1}$
$$f'(x) = -\frac{3x^2 + 6x + 4}{\left(x^3 + 3x^2 + 4x + 1\right)^2}$$
Since $f'(x) = 0$ has no real roots, the graph has no extrema.
$$f''(x) = \frac{12x^4 + 48x^3 + 78x^2 + 66x + 26}{\left(x^3 + 3x^2 + 4x + 1\right)^3}$$
The Critical number is $x = -0.316722$. $f''(x) > 0$ on $(-0.3176722, \infty)$ so the graph is concave up on this interval. $f''(x) < 0$ on $(-\infty, -0.3176722)$ so the graph is concave down on this interval. the graph has a vertical asymptote at $x = -0.3176722$. $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$.
Therefore, the graph has horizontal asymptote $y = 0$.



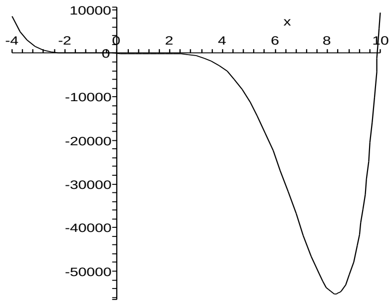
25. $f(x) = (x^3 - 3x^2 + 2x)^{2/3}$
$$f'(x) = \frac{2(3x^2 - 6x + 2)}{3(x^3 - 3x^2 + 2x)^{1/3}}$$
There are critical numbers at $x = \dfrac{3 \pm \sqrt{3}}{3}$, 0, 1 and 2.
$$f''(x) = \frac{18x^4 - 72x^3 + 84x^2 - 24x - 8}{9(x^3 - 3x^2 + 2x)^{4/3}}$$
with critical numbers $x = 0$, 1 and 2 and $x \approx -0.1883$ and 2.1883. $f''(x)$ changes sign at these last two values, so these are inflection points. The Second Derivative test shows that $x = \dfrac{3 \pm \sqrt{3}}{3}$ are both maxima. Local minima occur at $x = 0$, 1 and 2.
$f(x) \to \infty$ as $x \to \pm\infty$.



26. $f(x) = x^6 - 10x^5 - 7x^4 + 80x^3 + 12x^2 - 192x$
$f(x) \to \infty$ as $x \to \pm\infty$.
$f'(x) = 6x^5 - 50x^4 - 28x^3 + 240x^2 + 24x - 192$
Critical numbers at approximately $x = -1.9339, -1.0129, 1, 1.9644$, and 8.3158.
$f''(x) = 30x^4 - 200x^3 - 84x^2 + 480x + 24$
Critical numbers at approximately $x = -1.5534, -0.0496, 1.5430$, and 6.7267, and changes sign at each of these values, so these are inflection points. The Second Derivative Test shows that $x = -1.9339$, 1, and 8.3158 are local minima, and $x = -1.0129$ and 1.9644 are local maxima. The extrema near $x = 0$ look like this:



The inflection points, and the global behavior of the function can be seen on the following graph.



27. $f(x) = \dfrac{x^2+1}{3x^2-1}$
Note that $x = \pm\sqrt{1/3}$ are not in the domain of the function, but yield vertical asymptotes.

$$f'(x) = \frac{2x(3x^2-1)-(x^2+1)(6x)}{(3x^2-1)^2}$$
$$= \frac{(6x^3-2x)-(6x^3+6x)}{(3x^2-1)^2}$$
$$= \frac{-8x}{(3x^2-1)^2}$$

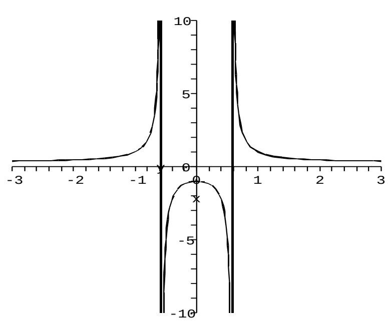
So the only critical point is $x = 0$.

$f'(x) > 0$ for $x < 0$
$f'(x) < 0$ for $x > 0$
so f is increasing on $(-\infty, -\sqrt{1/3})$ and on $(-\sqrt{1/3}, 0)$; decreasing on $(0, \sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$. Thus there is a local max at $x = 0$.

$$f''(x) = 8 \cdot \frac{9x^2+1}{(3x^2-1)^3}$$

$f''(x) > 0$ on $(-\infty, -\sqrt{1/3}) \cup (\sqrt{1/3}, \infty)$
$f''(x) < 0$ on $(-\sqrt{1/3}, \sqrt{1/3})$
Hence f is concave up on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$; concave down on $(-\sqrt{1/3}, \sqrt{1/3})$.

Finally, when $|x|$ is large, the function approached $1/3$, so $y = 1/3$ is a horizontal asymptote.



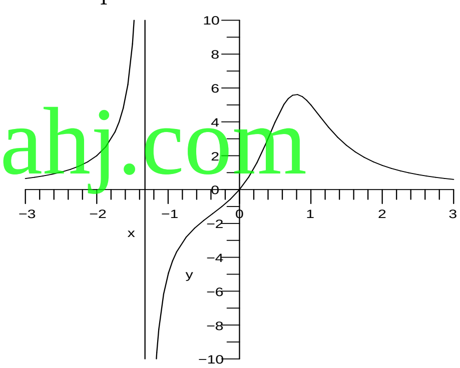
28. $f(x) = \dfrac{5x}{x^3-x+1}$
Looking at the graph of $x^3 - x + 1$, we see that there is one real root, at approximately -1.325; so the domain of the function is all x except for this one point, and $x = -1.325$ will be a vertical asymptote. There is a horizontal asymptote of $y = 0$.

$$f'(x) = 5\frac{1-2x^3}{(x^3-x-1)^2}$$

The only critical point is $x = \sqrt[3]{1/2}$. By the first derivative test, this is a local max.

$$f''(x) = 10\frac{3x^5+x^3-6x^2+1}{(x^3-x+1)^3}$$

The numerator of f'' has three real roots, which are approximately $x = -.39018$, $x = .43347$, and $x = 1.1077$. $f''(x) > 0$ on $(-\infty, -1.325) \cup (-.390, .433) \cup (1.108, \infty)$
$f''(x) < 0$ on $(-1.325, -.390) \cup (.433, 1.108)$
So f is concave up on $(-\infty, -1.325) \cup (-.390, .433) \cup (1.108, \infty)$ and concave down on $(-1.325, -.390) \cup (.433, 1.108)$. Hence $x = -.39018$, $x = .43347$, and $x = 1.1077$ are inflection points.



29. $f(x) = x^2\sqrt{x^2-9}$
f is undefined on $(-3, 3)$.

$$f'(x) = 2x\sqrt{x^2-9} + x^2\left(\frac{1}{2}(x^2-9)^{-1/2} \cdot 2x\right)$$
$$= 2x\sqrt{x^2-9} + \frac{x^3}{\sqrt{x^2-9}}$$
$$= \frac{2x(x^2-9)+x^3}{\sqrt{x^2-9}}$$
$$= \frac{3x^3-18x}{\sqrt{x^2-9}} = \frac{3x(x^2-6)}{\sqrt{x^2-9}}$$
$$= \frac{3x(x+\sqrt{6})(x-\sqrt{6})}{\sqrt{x^2-9}}$$

Critical points ± 3. (Note that f is undefined at $x = 0$, $\pm\sqrt{6}$.)

$$\begin{aligned} f''(x) &= \frac{(9x^2-18)\sqrt{x^2-9}}{x^2-9} \\ &\quad - \frac{(3x^3-18x)\cdot\frac{1}{2}(x^2-9)^{-1/2}\cdot 2x}{x^2-9} \\ &= \frac{(9x^2-18)(x^2-9)-x(3x^3-18x)}{(x^2-9)^{3/2}} \\ &= \frac{(6x^4-81x^2+162)}{(x^2-9)^{3/2}} \end{aligned}$$

$f''(x) = 0$ when

$$\begin{aligned} x^2 &= \frac{81 \pm \sqrt{81^2-4(6)(162)}}{2(6)} \\ &= \frac{81\pm\sqrt{2673}}{12} = \frac{1}{4}(27\pm\sqrt{297}) \end{aligned}$$

So $x \approx \pm 3.325$ or $x \approx \pm 1.562$, but these latter values are not in the same domain. So only ± 3.325 are potential inflection points.

$f'(x) > 0$ on $(3, \infty)$

$f'(x) < 0$ on $(-\infty, -3)$

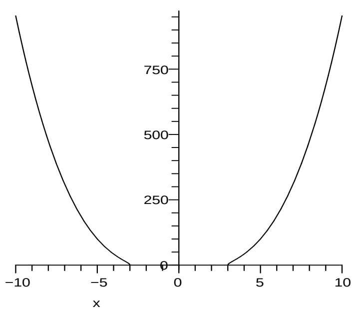
$f''(x) > 0$ on $(-\infty, -3.3) \cup (3.3, \infty)$

$f''(x) < 0$ on $(-3.3, -3) \cup (3, 3.3)$

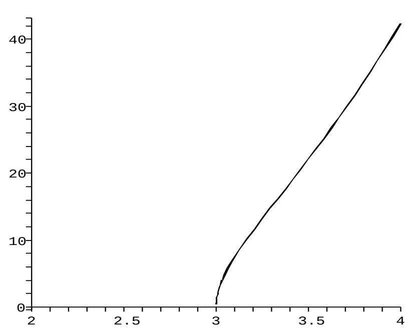
f is increasing on $(3, \infty)$, decreasing on $(-\infty, -3)$, concave up on $(-\infty, -3.3) \cup (3.3, \infty)$, concave down on $(-3.3, -3) \cup (3, 3.3)$.

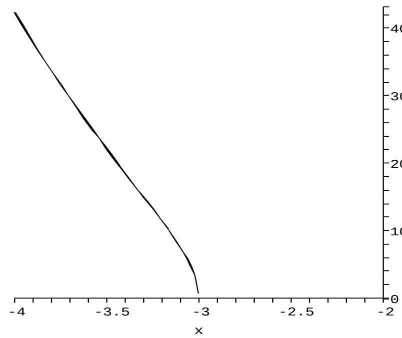
$x = \pm 3.3$ are inflection points.

Global graph of $f(x)$:



Local graphs of $f(x)$:





30. $f(x) = \sqrt[3]{2x^2-1}$

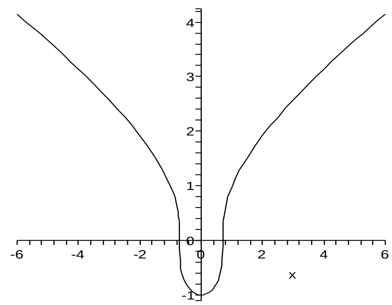
$f'(x) = \dfrac{4x}{3(2x^2-1)^{2/3}}$

$f'(x) = 0$ at $x = 0$ and is undefined at $x = \pm\sqrt{1/2}$.

$f''(x) = \dfrac{-4(2x^2+3)}{9(2x^2-1)^{5/3}}$

$f''(x)$ is never 0, and is undefined where f' is. The function changes concavity at $x = \pm\sqrt{1/2}$, so these are inflection points. The slope does not change at these values, so they are not extrema. The Second Derivative Test shows that $x = 0$ is a minimum.

$f(x) \to \infty$ as $x \to \pm\infty$.



31. $f(x) = e^{-2x}\sin x$

$f'(x) = e^{-2x}(\cos x - 2\sin x)$

$f''(x) = e^{-2x}(3\sin x - 4\cos x)$

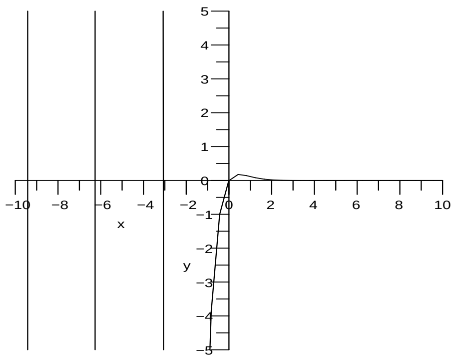
$f'(x) = 0$ when $\cos x = 2\sin x$; that is, when $\tan x = 1/2$; that is, when $x = k\pi + \tan^{-1}(1/2)$, where k is any integer.

$f'(x) < 0$, and f is decreasing, on intervals of the form $(2k\pi + \tan^{-1}(\frac{1}{2}), (2k+1)\pi + \tan^{-1}(\frac{1}{2}))$

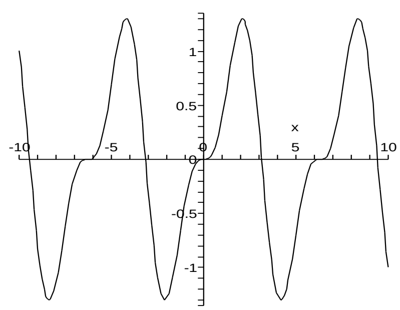
$f'(x) > 0$ and f is increasing, on intervals of the form $((2k-1)\pi + \tan^{-1}(\frac{1}{2}), 2k\pi + \tan^{-1}(\frac{1}{2}))$

Hence f has a local max at $x = 2k\pi + \tan^{-1}(1/2)$ and a local min at $x = (2k+1)\pi + \tan^{-1}(1/2)$.

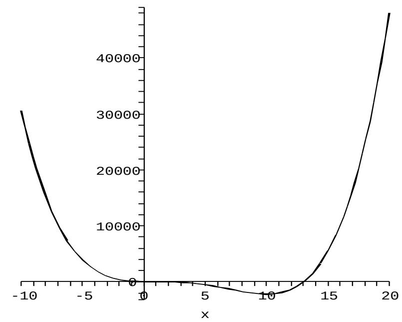
$f''(x) = 0$ when $3\sin x = 4\cos x$; that is, when $\tan x = 4/3$; that is, when $x = k\pi + \tan^{-1}(4/3)$. The sign of f'' changes at each of these points, so all of them are inflection points.



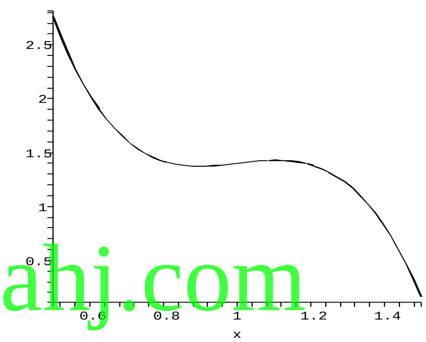
32. $f(x) = \sin x - \frac{1}{2}\sin 2x$
$f'(x) = \cos x - \cos 2x$
$f'(x) = 0$ when $x = 2k\pi$, $2\pi/3 + 2k\pi$, or $4\pi/3 + 2k\pi$.
$f''(x) = -\sin x + 2\sin 2x$
$f''(x) = 0$ when $x = 0$, π and approximately ± 1.3181, and the pattern repeats with period 2π. f'' changes sign at each of these values, so these are inflection points. The First Derivative Test shows that $x = 2k\pi$ is neither a minimum nor a maximum. The Second Derivative Test shows that the other critical numbers are extrema that alternate between minima and maxima.



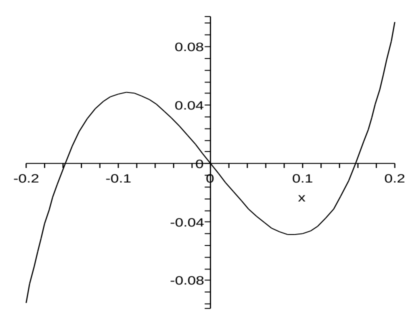
33. $f(x) = x^4 - 16x^3 + 42x^2 - 39.6x + 14$
$f'(x) = 4x^3 - 48x^2 + 84x - 39.6$
$f''(x) = 12x^2 - 96x + 84$
$= 12(x^2 - 8x + 7)$
$= 12(x - 7)(x - 1)$
$f'(x) > 0$ on $(.8952, 1.106) \cup (9.9987, \infty)$
$f'(x) < 0$ on $(-\infty, .8952) \cup (1.106, 9.9987)$
$f''(x) > 0$ on $(-\infty, 1) \cup (7, \infty)$
$f''(x) < 0$ on $(1, 7)$
f is increasing on $(.8952, 1.106)$ and on $(9.9987, \infty)$, decreasing on $(-\infty, .8952)$ and on $(1.106, 9.9987)$, concave up on $(-\infty, 1) \cup (7, \infty)$, concave down on $(1, 7)$, $x = .8952$, 9.9987 are local min, $x = 1.106$ is local max, $x = 1, 7$ are inflection points.
$f(x) \to \infty$ as $x \to \pm\infty$.
Global graph of $f(x)$:



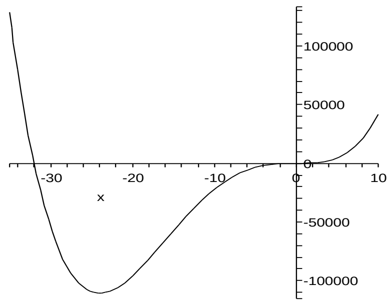
Local graph of $f(x)$:

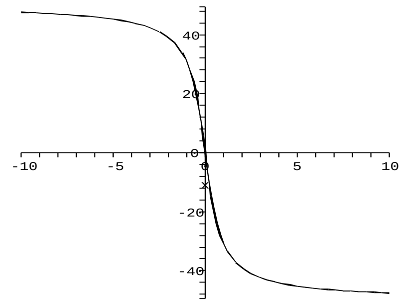


34. $f(x) = x^4 + 32x^3 - 0.02x^2 - 0.8x$
$f'(x) = 4x^3 + 96x^2 - 0.04x - 0.8$
$f'(x) = 0$ at approximately $x = -24$, -0.09125, and 0.09132.
$f''(x) = 12x^2 + 192x - 0.04$
$f''(x) = 0$ at approximately $x = 16.0002$ and 0.0002, and changes sign at these values, so these are inflection points. The Second Derivative Test shows that $x = -24$ and 0.09132 are minima, and that $x = -0.09125$ is a maxima. The extrema near $x = 0$ look like this:



The global behavior looks like this:





35. $f(x) = \dfrac{25 - 50\sqrt{x^2 + 0.25}}{x}$

$$= 25\left(\frac{1 - 2\sqrt{x^2 + 0.25}}{x}\right)$$

$$= 25\left(\frac{1 - \sqrt{4x^2 + 1}}{x}\right)$$

Note that $x = 0$ is not in the domain of the function.

$f'(x) = 25\left(\dfrac{1 - \sqrt{4x^2 + 1}}{x^2\sqrt{4x^2 + 1}}\right)$

We see that there are no critical points. Indeed, $f' < 0$ wherever f is defined. One can verify that

$f''(x) > 0$ on $(0, \infty)$
$f''(x) < 0$ on $(-\infty, 0)$
Hence the function is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

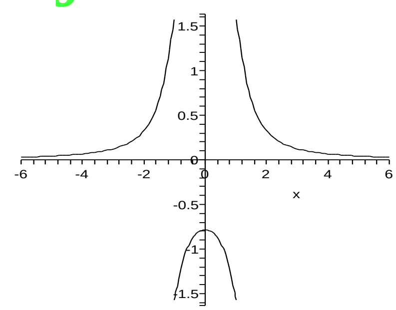
$$\lim_{x\to\infty} \frac{25 - 50\sqrt{x^2 + 0.25}}{x}$$
$$= \lim_{x\to\infty} \frac{25}{x} - \frac{50\sqrt{x^2 + 0.25}}{x}$$
$$= \lim_{x\to\infty} 0 - 50\frac{x\sqrt{1 + \frac{0.25}{x^2}}}{x}$$
$$= \lim_{x\to\infty} -50\sqrt{1 + \frac{0.25}{x^2}} = -50$$

$$\lim_{x\to-\infty} \frac{25 - 50\sqrt{x^2 + 0.25}}{x}$$
$$= \lim_{x\to\infty} \frac{25}{x} - \frac{50\sqrt{x^2 + 0.25}}{x}$$
$$= \lim_{x\to-\infty} 0 - 50\frac{(-x)\sqrt{1 + \frac{0.25}{x^2}}}{x}$$
$$= \lim_{x\to\infty} 50\sqrt{1 + \frac{0.25}{x^2}} = 50$$

So f has horizontal asymptotes at $y = 50$ and $y = -50$.

36. $f(x) = \tan^{-1}\left(\dfrac{1}{x^2 - 1}\right)$

The function has horizontal asymptote $y = 0$, and is undefined at $x = \pm 1$.
$f'(x) = \dfrac{-2x}{x^4 - 2x^2 + 2}$
$f'(x) = 0$ only when $x = 0$.
$f''(x) = \dfrac{2(3x^4 - 2x^2 - 2)}{(x^4 - 2x^2 + 2)^2}$
$f''(x) = 0$ at approximately $x = \pm 1.1024$ and changes sign there, so these are inflection points (very easy to miss by looking at the graph). The Second Derivative Test shows that $x = 0$ is a local maximum.



37. $f(x) = x^4 + cx^2$
$f'(x) = 4x^3 + 2cx$
$f''(x) = 12x^2 + 2c$
$c = 0$: 1 extremum, 0 inflection points
$c < 0$: 3 extrema, 2 inflection points
$c > 0$: 1 extremum, 0 inflection points
$c \to -\infty$: the graph widens and lowers
$c \to +\infty$: the graph narrows

38. $f(x) = x^4 + cx^2 + x$
$f'(x) = 4x^3 + 2cx + 1$
$f''(x) = 12x^2 + 2c$
If c is negative, there will be two solutions to $f'' = 0$, and these will be inflection points. For $c > 0$ there will be no solutions to $f'' = 0$, and no inflection points. For $c = 0$, $f'' = 0$

when $x = 0$, but does not change sign there, so this is not an inflection point. $f' = 0$ has one solution, corresponding to a minimum, for all $c > -1.5$. For $c = -1.5$, there is a second critical point which is neither a minimum nor a maximum. For $c < -1.5$ there are three critical points, two minima and a maximum. As $c \to \infty$ the curve has one minimum, and narrows. As $c \to -\infty$, the two minima get farther apart and drop lower. The local maximum approaches $(0, 0)$.

39. $f(x) = \dfrac{x^2}{x^2 + c^2}$

$f'(x) = \dfrac{2c^2 x}{(x^2 + c^2)^2}$

$f''(x) = \dfrac{2c^4 - 6c^2x^2}{(x^2 + c^2)^3}$

If $c = 0$: $f(x) = 1$, except that f is undefined at $x = 0$. $c < 0$, $c > 0$: horizontal asymptote at $y = 1$, local min at $x = 0$, since the derivative changes sign from negative to positive at $x = 0$; also there are inflection points at $x = \pm c/\sqrt{3}$. As $c \to -\infty$, $c \to +\infty$: the graph widens.

40. $f(x) = e^{-x^2/c}$

$f'(x) = \dfrac{-2x}{c} \cdot e^{-x^2/c}$

$f''(x) = \dfrac{-2c + 4x^2}{c^2} \cdot e^{-x^2/c}$

For $c > 0$ the graph is a bell curve centered at its maximum point $(0, 1)$, and the inflection points are at $x = \pm\sqrt{c/2}$. As $c \to \infty$, the curve widens.

The function is not defined for $c = 0$.

For $c < 0$, there are no inflection points, and $x = 0$ is a minimum. The graph is cup shaped and widens as $c \to -\infty$.

41. When $c = 0$, $f(x) = \sin(0) = 0$.

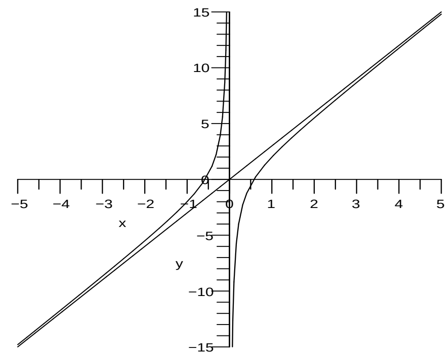
Since $\sin x$ is an odd function, $\sin(-cx) = -\sin(cx)$. Thus negative values of c give the reflection through the x-axis of their positive counterparts. For large values of c, the graph looks just like $\sin x$, but with a very small period.

42. When $c = 0$, we have $f(x) = x^2\sqrt{-x^2}$, which is undefined.

Since $x^2\sqrt{c^2 - x^2} = x^2\sqrt{(-c)^2 - x^2}$, the function is the same regardless of whether c is negative or positive. The function is always 0 at $x = 0$ and undefined for $|x| > |c|$. Where it is defined, $f(x) \geq 0$, attaining its minimum at $x = 0$. It reaches its maximum value at $x = \pm\sqrt{2c^2/3}$. At these points, f attains the value $2\sqrt{3}|c|^3/9$. The function looks generally the same as $|c|$ gets large, with the domain and range increasing as $|c|$ does.

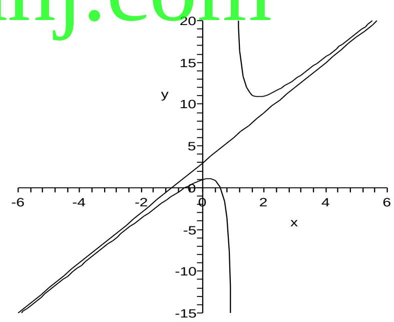
43. $f(x) = \dfrac{3x^2 - 1}{x} = 3x - \dfrac{1}{x}$

$y = 3x$ is a slant asymptote.



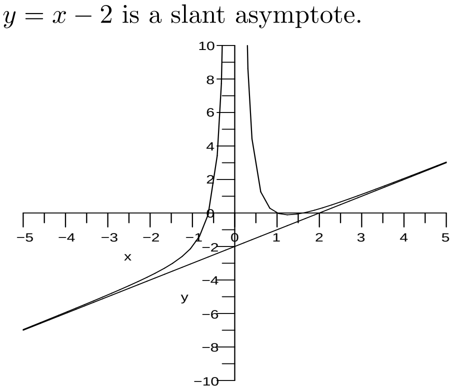
44. $f(x) = \dfrac{3x^2 - 1}{x - 1} = 3x + 3 + \dfrac{2}{x - 1}$,

so the slant asymptote is $y = 3x + 3$.

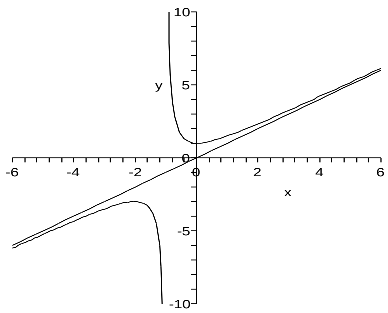


45. $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2} = x - 2 + \dfrac{1}{x^2}$

$y = x - 2$ is a slant asymptote.

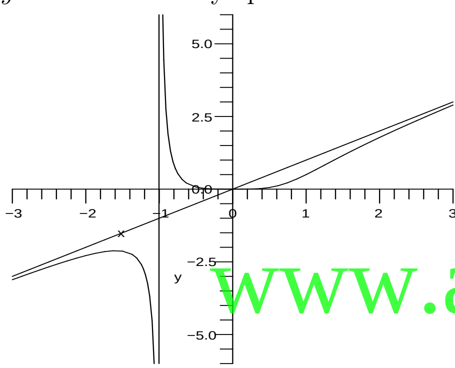


46. $f(x) = \dfrac{x^3 - 1}{x^2 - 1} = x + \dfrac{x - 1}{x^2 - 1}$, so the slant asymptote is $y = x$.



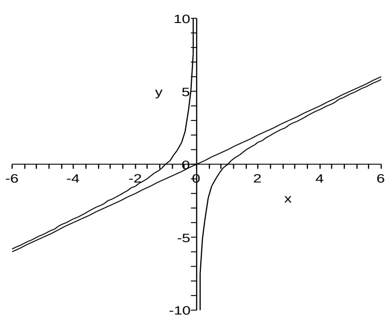
47. $f(x) = \dfrac{x^4}{x^3+1} = x - \dfrac{x}{x^3+1}$

$y = x$ is a slant asymptote.



48. $f(x) = \dfrac{x^4-1}{x^3+x} = x + \dfrac{-x^2-1}{x^3+x}$,

so the slant asymptote is $y = x$.



49. One possibility:

$$f(x) = \frac{3x^2}{(x-1)(x-2)}$$

50. One possibility:

$$f(x) = \frac{x}{x^2-1}$$

51. One possibility:

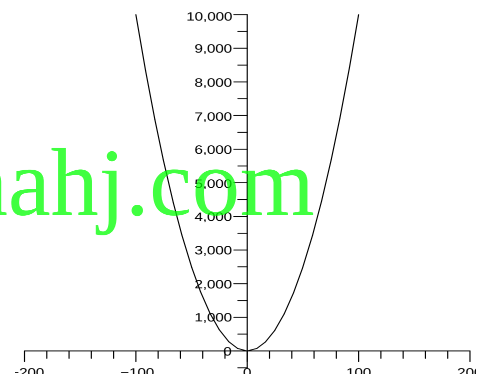
$$f(x) = \frac{2x}{\sqrt{(x-1)(x+1)}}$$

52. One possibility:

$$f(x) = \frac{2x^2}{(x-1)(x-3)}$$

53. $$\lim_{x\to\infty}\left[\frac{x^4-x^2+1}{x^2-1} - x^2\right]$$
$$= \lim_{x\to\infty}\left[\frac{x^4-x^2+1-x^2(x^2-1)}{x^2-1}\right]$$
$$= \lim_{x\to\infty}\left[\frac{1}{x^2-1}\right] = 0$$

Thus $f(x) = \dfrac{x^4-x^2+1}{x^2-1}$ has x^2 as an asymptote.



54. (a) $$f(x) = \frac{x^4}{x+1}$$
$$= \frac{x^4-1+1}{x+1}$$
$$= \frac{(x^2+1)(x+1)(x-1)}{x-1} + \frac{1}{x+1}$$
$$= (x^2+1)(x+1) + \frac{1}{x+1}$$

One possible polynomial is $p(x) = (x^2+1)(x+1)$. Then $|f(x) - p(x)| = \left|\frac{1}{x+1}\right| \to 0$ as $x \to \infty$.

(b) $$f(x) = \frac{x^5-1}{x+1}$$
$$= \frac{x^5+1-2}{x+1}$$
$$= x^4 - x^3 + x^2 - x + 1 - \frac{2}{x+1}$$

One possible polynomial is $p(x) = x^4 - x^3 + x^2 - x + 1$. Then $|f(x) - p(x)| = \left|\frac{2}{x+1}\right| \to 0$ as $x \to \infty$.

(c) $f(x) = \dfrac{x^6-2}{x+1}$
$= \dfrac{x^6-1-1}{x+1}$
$= \dfrac{(x^3-1)(x^3+1)-1}{x+1}$
$= \dfrac{(x^3-1)(x+1)(x^2-x+1)-1}{x+1}$
$= (x^3-1)(x^2-x+1) - \dfrac{1}{x+1}$

One possible polynomial is $p(x) = (x^3-1)(x^2-x+1)$. Then $|f(x)-p(x)| = \left|\frac{1}{x+1}\right| \to 0$ as $x \to \infty$.

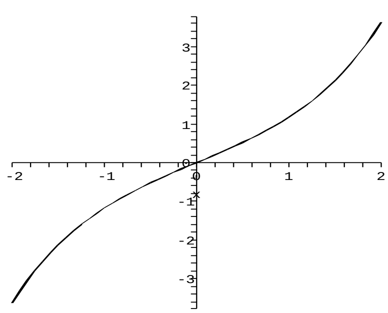
55. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$f'(x) = \dfrac{e^x + e^{-x}}{2}$

$f'(x) > 0$ for all x so $f(x)$ is always increasing and has no extrema.

$f''(x) = \dfrac{e^x - e^{-x}}{2}$

$f''(x) = 0$ only when $x = 0$ and changes sign here, so $f(x)$ has an inflection point at $x = 0$.



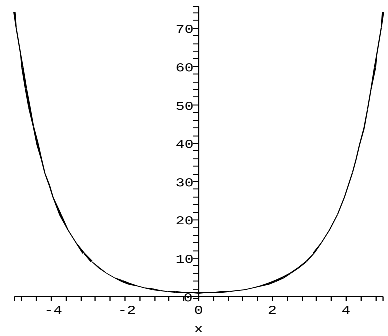
$f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$f'(x) = \dfrac{e^x - e^{-x}}{2}$

$f'(x) = 0$ only when $x = 0$.

$f''(x) = \dfrac{e^x + e^{-x}}{2}$

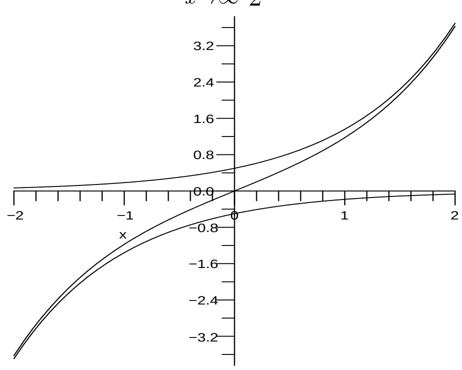
$f''(x) > 0$ for all x, so $f(x)$ has no inflection points, but $x = 0$ is a minimum.



56. For $y = \sinh x$ we need to use $-\frac{1}{2}e^{-x}$ instead of $\frac{1}{2}e^{-x}$. To explain the enveloping behavior, note that:

$$\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} = \lim_{x\to-\infty} -\frac{e^{-x}}{2}$$

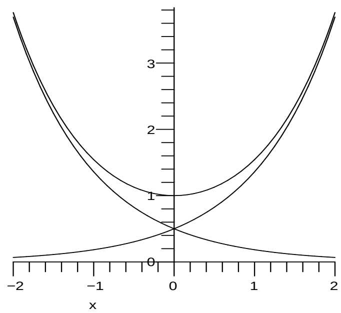
$$\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \lim_{x\to\infty} \frac{e^x}{2}$$



To explain the enveloping behavior for $y = \cosh x$, note that:

$$\lim_{x\to-\infty} \cosh x = \lim_{x\to-\infty} \frac{e^x + e^{-x}}{2} = \lim_{x\to-\infty} \frac{e^{-x}}{2}$$

$$\lim_{x\to\infty} \cosh x = \lim_{x\to\infty} \frac{e^x + e^{-x}}{2} = \lim_{x\to-\infty} \frac{e^x}{2}$$



57. $f(x) = xe^{-bx}$
$f(0) = 0$
$f(x) > 0$ for $x > 0$
$\lim\limits_{x\to\infty} xe^{-bx} = \lim\limits_{x\to\infty} \dfrac{x}{e^{bx}} = \lim\limits_{x\to\infty} \dfrac{1}{be^{bx}} = 0$
(by L'Hôpital's rule)
$f'(x) = e^{-bx}(1 - bx)$, so there is a unique critical point at $x = 1/b$, which must be the maximum. The bigger b is, the closer the max is to the origin. For time since conception, $1/b$ represents the most common gestation time. For survival time, $1/b$ represents the most common life span.

58. From the graph we can count 15 maxima and 16 minima in the range $0 \le x \le 10$. Using a CAS to solve
$f'(x) = -\sin(10x + 2\cos x)(10 - 2\sin x) = 0$,
we find the following values of x at the extrema.

Minima	Maxima
0.11549	0.44806
0.80366	1.18055
1.57080	1.96104
2.33793	2.69353
3.02610	3.33776
3.63216	3.91326
4.18477	4.45009
4.71239	7.97469
5.24001	5.51152
5.79261	6.08702
6.39868	6.73125
7.08685	7.46374
7.85398	8.24422
8.62112	8.97672
9.30929	9.62094
9.91535	

59. $f(x) = \mathrm{A} = \tan^{-1}\left(\dfrac{29.25}{x}\right) - \tan^{-1}\left(\dfrac{10.75}{x}\right)$

$$f'(x) = \left[\frac{x^2}{x^2 + (29.25)^2}\left(\frac{-29.25}{x^2}\right)\right] - \left[\frac{x^2}{x^2 + (10.75)^2}\left(\frac{-10.75}{x^2}\right)\right]$$

$f'(x) = 0 \Rightarrow$
$$\frac{-29.25}{x^2 + (29.25)^2} + \frac{10.75}{x^2 + (10.75)^2} = 0$$
$x = 17.73$ ft.
Substitue $x = 17.73$ in $f(x)$.
$$A = \tan^{-1}\left(\frac{29.25}{17.73}\right) - \tan^{-1}\left(\frac{10.75}{17.73}\right)$$
$= 58.78 - 31.23$
$= 27.55°$

Now x is increased to $(x + 15)$.
$f(x + 15) = \mathrm{A}$
$$= \tan^{-1}\left(\frac{29.25}{x+15}\right) - \tan^{-1}\left(\frac{10.75}{x+15}\right)$$
$f'(x + 15)$
$$= \left[\frac{1}{1 + \left(\frac{29.25}{x+15}\right)^2}\left(\frac{-29.25}{(x+15)^2}\right)\right] - \left[\frac{1}{1 + \left(\frac{10.75}{x+15}\right)^2}\left(\frac{-10.75}{(x+15)^2}\right)\right]$$
$$= \frac{-29.25}{(x+15)^2 + (29.25)^2} + \frac{10.75}{(x+15)^2 + (10.75)^2}$$

$f'(x) = 0 \Rightarrow x = 2.73$ ft.
Substitute $x = 2.73$ in $f(x)$.
$$A = \tan^{-1}\left(\frac{29.25}{2.73}\right) - \tan^{-1}\left(\frac{10.75}{2.73}\right)$$
$= 84.67 - 75.75$
$= 8.92°$
Therefore, A decreases by $18.63°$.

60. $x(t) = \dfrac{2.5}{w}t - \dfrac{2.5}{4w^2}\sin 4wt$
Since $0 \le t \le 0.68$. Hence
$$\begin{aligned} 0 \le x(t) &\le \left(\tfrac{(2.5)(0.68)}{w} - \tfrac{2.5}{4w^2}\sin(4w(0.68))\right) \\ &\le \left(\tfrac{1.7}{w} - \tfrac{2.5}{4w^2}\sin(2.72w)\right) \\ &\le \left(\tfrac{6.8w - 2.5\sin(2.72w)}{4w^2}\right) \end{aligned}$$
Taking limit as $w \to 0$
$$\lim_{w\to 0} 0 \le \lim_{w\to 0} x(t) \le \lim_{w\to 0}\left(\tfrac{6.8w - 2.5\sin(2.72w)}{4w^2}\right)$$
(by L'Hôpital's rule)

$\lim_{w\to 0}\left(\frac{6.8w - 2.5\sin(2.72w)}{4w^2}\right)$
$= \lim_{w\to 0}\frac{6.8 - 6.8\cos(2.72w)}{8w}$
(by L'Hôpital's rule)
$\lim_{w\to 0}\frac{18.496\sin(2.72w)}{8} = 0$
Hence
$\lim_{w\to 0} 0 \le \lim_{w\to 0} x(t) \le 0.$
As $w \to 0$, $x(t) \to 0$ that is the knuckleball will move in a straight direction.

3.7 Optimization

1. $A = xy = 1800$
$y = \frac{1800}{x}$
$P = 2x + y = 2x + \frac{1800}{x}$
$P' = 2 - \frac{1800}{x^2} = 0$
$2x^2 = 1800$
$x = 30$
$P'(x) > 0$ for $x > 30$
$P'(x) < 0$ for $0 < x < 30$
So $x = 30$ is min.

$y = \frac{1800}{x} = \frac{1800}{30} = 60$
So the dimensions are $30' \times 60'$ and the minimum perimeter is 120 ft.

2. If y is the length of fence opposite the river, and x is the length of the other two sides, then we have the constraint $2x + y = 96$. We wish to maximize
$A = xy = x(96 - 2x)$.
$A' = 96 - 4x = 0$ when $x = 24$.
$A'' = -4 < 0$ so this gives a maximum. Reasonable possible values of x range from 0 to 48, and the area is 0 at these extremes. The maximum area is $A = 1152$, and the dimensions are $x = 24$, $y = 48$.

3. $P = 2x + 3y = 120$
$3y = 120 - 2x$
$y = 40 - \frac{2}{3}x$
$A = xy$

$$A(x) = x\left(40 - \frac{2}{3}x\right)$$
$$A'(x) = 1\left(40 - \frac{2}{3}x\right) + x\left(-\frac{2}{3}\right)$$
$$= 40 - \frac{4}{3}x = 0$$
$$40 = \frac{4}{3}x$$
$$x = 30$$
$A'(x) > 0$ for $0 < x < 30$
$A'(x) < 0$ for $x > 30$.
So $x = 30$ is max, $y = 40 - \frac{2}{3}\cdot 30 = 20$.
So the dimensions are $20' \times 30'$.

4. Let x be the length of the sides facing each other and y be the length of the third side. We have the constraint that $xy = 800$, or $y = 800/x$. We also know that $x > 6$ and $y > 10$. The function we wish to minimize is the length of walls needed, or the side length minus the width of the doors.
$L = (y - 10) + 2(x - 6) = 800/x + 2x - 22$.
$L' = -800/x^2 + 2 = 0$ when $x = 20$.
$L'' = 1600/x^3 > 0$ when $x = 20$ so this is a minimum. Possible values of x range from 6 to 80. $L(6) = 123.3$, $L(80) = 148$, and $L(20) = 58$. To minimize the length of wall, the facing sides should be 20 feet, and the third side should be 40 feet.

5.
$$A = xy$$
$$P = 2x + 2y$$
$$2y = P - 2x$$
$$y = \frac{P}{2} - x$$
$$A(x) = x\left(\frac{P}{2} - x\right)$$
$$A'(x) = 1\cdot\left(\frac{P}{2} - x\right) + x(-1)$$
$$= \frac{P}{2} - 2x = 0$$
$$P = 4x$$
$$x = \frac{P}{4}$$
$A'(x) > 0$ for $0 < x < P/4$
$A'(x) < 0$ for $x > P/4$

So $x = P/4$ is max,
$y = \frac{P}{2} - x = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$
So the dimensions are $\frac{P}{4} \times \frac{P}{4}$. Thus we have a square.

6. We have a rectangle with sides x and y and area $A = xy$, and that we wish to minimize the perimeter,
$P = 2x + 2y = 2x + 2 \cdot \dfrac{A}{x}$.
$P' = 2 - \dfrac{2A}{x^2} = 0$ when $x = \sqrt{A}$.
$P'' = 4A/x^3 > 0$ here, so this is a minimum. Possible values of x range from 0 to ∞. As x approaches these values the perimeter grows without bound. For fixed area, the rectangle with minimum perimeter has dimensions $x = y = \sqrt{A}$, a square.

7. $V = l \cdot w \cdot h$
$V(x) = (10 - 2x)(6 - 2x) \cdot x,\ 0 \le x \le 3$

$$\begin{aligned} V'(x) &= -2(6-2x)\cdot x + (10-2x)(-2)\cdot x \\ &\quad + (10-2x)(6-2x) \\ &= 60 - 64x + 12x^2 \\ &= 4(3x^2 - 16x + 15) \\ &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{16 \pm \sqrt{(-16)^2 - 4\cdot 3\cdot 15}}{6} \\ &= \frac{8}{3} \pm \frac{\sqrt{19}}{3} \\ x &= \frac{8}{3} + \frac{\sqrt{19}}{3} > 3. \end{aligned}$$

$V'(x) > 0$ for $x < 8/3 - \sqrt{19}/3$
$V'(x) < 0$ for $x > 8/3 - \sqrt{19}/3$

So $x = \dfrac{8}{3} - \dfrac{\sqrt{19}}{3}$ is a max.

8. If we cut squares out of the corners of a 12" by 16" sheet and fold it into a box, the volume of the resulting box will be

$$\begin{aligned} V &= x(12-2x)(16-2x) \\ &= 4x^3 - 56x^2 + 192x, \end{aligned}$$

where the value of x must be between 0 and 6.
$V' = 12x^2 - 112x + 192 = 0$
when $x = \frac{14 \pm 2\sqrt{13}}{3} \approx 7.07$ and 2.26. The critical value $x = \frac{14+2\sqrt{13}}{3}$ is outside of the reasonable range. The volume is 0 when x is 0 or 6. The First Derivative Test shows that $x = \frac{14-2\sqrt{13}}{3}$ gives the maximum volume.

9. (a) $V = l \cdot w \cdot h$
The volume of the first box (without top) is
$V_1 = V_1(x) = (6-2x)^2(x) = 4x(3-x)^2$ where $0 < x < 3$. The volume of the second box (without top and bottom) is
$V_2 = V_2(x) = x^3$.
Thus, we find the absolute maximum of the continuous function
$V = V(x) = V_1(x) + V_2(x) = 4x(3-x)^2 + x^3$
on the interval $0 < x < 3$.

$$\begin{aligned} V'(x) &= 4(3-x)^2 + 4x\,(2\,(3-x)\,(-1)) + 3x^2 \\ &= 4\left(9 - 6x + x^2\right) - 8x\,(3-x) + 3x^2 \\ &= 15x^2 - 48x + 36 \\ &= (x-2)\,(15x-18) \end{aligned}$$

Now compare the value of the function at the critical points.
$V(1.2) = 17.28$
$V(2) = 16$
Therefore, the value $x = 1.2$ maximizes the sum of volumes of the boxes.

(b) The volume of the first box (without top) is
$V_1 = V_1(x) = (6-2x)(4-2x)(x)$
$= 4x(3-x)(2-x)$, where $0 < x < 2$
The volume of the second box (without top and bottom) is
$V_2 = V_2(x) = x^3$.
Thus, we find the absolute maximum of the continuous function
$V = V(x) = V_1(x) + V_2(x)$
$= 4x(3-x)(2-x) + x^3$, on the interval $0 < x < 2$.
We have,

$$\begin{aligned} V'(x) &= 4\,(3-x)\,(2-x) + 4x\,(2-x)\,(-1) \\ &\quad + 4x\,(3-x)\,(-1) + 3x^2 \\ &= 4\left(6 - 5x + x^2\right) - 4x\,(2-x) \\ &\quad - 4x\,(3-x) + 3x^2 \\ &= 15x^2 - 40x + 24 \end{aligned}$$

Now compare the value of the function at the critical points.
$V(0.91169) = 9.0$
$V(1.75496) = 5.4$
Therefore, $x = 0.91169$ maximizes the sum of volumes of the boxes.

10. The volume of the first box (without top) is
$V_1 = V_1(x) = (6-2x)(d-2x)(x)$,
where $0 < x < \min\{\frac{d}{2}, 3\}$
The volume of the second box (without top and bottom) is
$V_2 = V_2(x) = x^3$.
Thus, we find the absolute maximum of the continuous function
$V = V(x) = V_1(x) + V_2(x)$
$= x(6-2x)(d-2x) + x^3$, on the interval $0 < x < \min\{\frac{d}{2}, 3\}$
We have,

$$\begin{aligned}V'(x) &= (-2x)(d-2x) - 2x(6-2x)\\ &\quad + (d-2x)(6-2x) + 3x^2\\ &= 15x^2 - 24x - 4dx + 6d\\ &= 15x^2 - x(24+4d) + 6d = 0\end{aligned}$$

To get real values of x, $(24+4d)^2 - 4\cdot 15\cdot 6d \geq 0$ and this is ture for all $d > 0$.
Therefore for each $d > 0$, we can find x such that V is maximum.

11. $d = \sqrt{(x-0)^2 + (y-1)^2}$
$y = x^2$
$$\begin{aligned}d &= \sqrt{x^2 + (x^2-1)^2}\\ &= (x^4 - x^2 + 1)^{1/2}\\ d'(x) &= \frac{1}{2}(x^4 - x^2 + 1)^{-1/2}(4x^3 - 2x)\\ &= \frac{2x(2x^2-1)}{2\sqrt{x^4 - x^2 + 1}} = 0\end{aligned}$$
$x = 0, \pm\sqrt{1/2}$;
$f(0) = 1$, $f(\sqrt{1/2}) = 3/4$, $f(-\sqrt{1/2}) = \frac{3}{4}$;
Thus $x = \pm\sqrt{1/2}$ are min, and the points on $y = x^2$ closest to $(0,1)$ are $(\sqrt{1/2}, 1/2)$ and $(-\sqrt{1/2}, 1/2)$.

12. Points on the curve $y = x^2$ can be written (x, x^2). The distance from such a point to $(3,4)$ is
$$\begin{aligned}D &= \sqrt{(x-3)^2 + (x^2-4)^2}\\ &= \sqrt{x^4 - 7x^2 - 6x + 25}.\end{aligned}$$
We numerically approximate the solution of $D' = \dfrac{2x^3 - 7x - 3}{\sqrt{x^4 - 7x^2 - 6x + 25}} = 0$ to be $x \approx 2.05655$, and two negative solutions. The negative critical numbers clearly do not minimize the distance. The closest point is approximately $(2.05655, 4.22940)$.

13.
$$\begin{aligned}d &= \sqrt{(x-0)^2 + (y-0)^2}\\ y &= \cos x\\ d &= \sqrt{x^2 + \cos^2 x}\\ d'(x) &= \frac{2x - 2\cos x \sin x}{2\sqrt{x^2 + \cos^2 x}} = 0\\ x &= \cos x \sin x\\ x &= 0\end{aligned}$$
So $x = 0$ is min and the point on $y = \cos x$ closest to $(0,0)$ is $(0,1)$.

14. Points on the curve $y = \cos x$ can be written $(x, \cos x)$. The distance from such a point to $(1,1)$ is
$$\begin{aligned}D &= \sqrt{(x-1)^2 + (\cos x - 1)^2}\\ &= \sqrt{x^2 - 2x + \cos^2 x - 2\cos x + 2}\end{aligned}$$
We numerically approximate the solution of
$$\begin{aligned}D' &= \frac{x - 1 - \cos x \sin x + \sin x}{\sqrt{x^2 - 2x + \cos^2 x - 2\cos x + 2}}\\ &= 0\end{aligned}$$
to be $x \approx 0.789781$. The First or Second Derivative Test shows that this is a minimum distance. The closest point is approximately $(0.789781, 0.704001)$.

15. For $(0,1)$, $(\sqrt{1/2}, 1/2)$ on $y = x^2$, we have $y' = 2x$, $y'(\sqrt{1/2}) = 2\cdot\sqrt{1/2} = \sqrt{2}$ and
$$m = \frac{\frac{1}{2} - 1}{-\sqrt{\frac{1}{2}} - 0} = \frac{1}{\sqrt{2}}.$$
For $(0,1)$, $(-\sqrt{1/2}, 1/2)$ on $y = x^2$, we have $y'(-\sqrt{1/2}) = 2(-\sqrt{1/2}) = -\sqrt{2}$ and
For $(3,4)$, $(2.06, 4.2436)$ on $y = x^2$, we have $y'(2.06) = 2(2.06) = 4.12$ and
$$m = \frac{4.2436 - 4}{2.06 - 3} = -0.2591 \approx -\frac{1}{4.12}.$$

16. For $(3,9)$, $(1,8)$ on $y = 9 - x^2$, we have $y' = -2x$, $y'(1) = -2\cdot 1 = -2$ and
$$m = \frac{8-9}{1-3} = \frac{1}{2}.$$
For $(5,11)$, $(0.79728, 8.364)$ on $y = 9 - x^2$, we have $y'(0.79728) = -2(0.79728) = -1.59456$ and
$$m = \frac{8.364 - 11}{0.79728 - 5} = 0.6272 \approx \frac{1}{1.59456}.$$

17. Cost: $C = 2(2\pi r^2) + 2\pi r h$
Convert from fluid ounces to cubic inches:
$$\begin{aligned}12 \text{ fl oz } &= 12 \text{ fl oz } \cdot 1.80469 \text{ in}^3/\text{fl oz}\\ &= 21.65628 \text{ in}^3\end{aligned}$$
Volume: $V = \pi r^2 h$ so
$$\begin{aligned}h &= \frac{V}{\pi r^2} = \frac{21.65628}{\pi r^2}\\ C &= 4\pi r^2 + 2\pi r\left(\frac{21.65628}{\pi r^2}\right)\\ C(r) &= 4\pi r^2 + 43.31256 r^{-1}\\ C'(r) &= 8\pi r - 43.31256 r^{-2}\\ &= \frac{8\pi r^3 - 43.31256}{r^2}\\ r &= \sqrt[3]{\frac{43.31256}{8\pi}} = 1.1989''\end{aligned}$$
when $C'(r) = 0$.
$C'(r) < 0$ on $(0, 1.1989)$
$C'(r) > 0$ on $(1.1989, \infty)$
Thus $r = 1.1989$ minimizes the cost and
$$h = \frac{21.65628}{\pi(1.1989)^2} = 4.7957''.$$

18. If the top and bottom of the cans are 2.23 times as thick as the sides, then the new cost func-

tion will be
$C(r) = 2\pi\left(2.23r^2 + \frac{21.65628}{\pi r}\right)$.
Then $C'(r) = 2\pi\left(4.46r - \frac{21.65628}{\pi r^2}\right) = 0$
when $r = \sqrt[3]{\frac{21.65628}{4.46\pi}} \approx 1.156$.
The First Derivative Test shows this is a minimum, and we can verify that the cost increases without bound as $r \to 0$ and $r \to \infty$.

19. Let x be the distance from the connection point to the easternmost development. Then $0 \le x \le 5$.

$$f(x) = \sqrt{3^2 + (5-x)^2} + \sqrt{4^2 + x^2},\quad 0 \le x \le 5$$

$$f'(x) = -(9 + (5-x)^2)^{-1/2}(5-x) + \frac{1}{2}(16 + x^2)^{-1/2}(2x) = \frac{x-5}{\sqrt{9+(5-x)^2}} + \frac{x}{\sqrt{16+x^2}} = 0$$

$$x = \frac{20}{7} \approx 2.857$$

$f(0) = 4 + \sqrt{34} \approx 9.831$

$f\left(\frac{20}{7}\right) = \sqrt{74} \approx 8.602$

$f(5) = 3 + \sqrt{41} \approx 9.403$

So $x = 20/7$ is minimum. The length of new line at this point is approximately 8.6 miles. Since $f(0) \approx 9.8$ and $f(5) \approx 9.4$, the water line should be 20/7 miles west of the second development.

20. Say the pipeline intersects the shore at a distance x from the closest point on the shore to the oil rig. Then x will be between 0 and 8. The length of underwater pipe is then $W = \sqrt{x^2 + 25^2}$, and the length of pipe constructed on land will be $L = \sqrt{(8-x)^2 + 5^2}$. The total cost will be $C = 50W + 20L$.

We numerically solve
$C' = \frac{50x}{\sqrt{625 + x^2}} + \frac{10(2x-16)}{\sqrt{x^2 - 16x + 89}} = 0$
to find $x \approx 5.108987$. The first derivative test shows that this gives a minimum. The cost at this value is \$1391 thousand. The cost when $x = 0$ is \$1439 thousand, and the cost when $x = 8$ is \$1412 thousand, so $x = 5.108987$ gives the absolute minimum cost.

21. (a) $C(x) = 5\sqrt{16 + x^2} + 2\sqrt{36 + (8-x)^2}$
$0 \le x \le 8$

$C(x) = 5\sqrt{16 + x^2} + 2\sqrt{100 - 16x + x^2}$

$$C'(x) = 5\left(\frac{1}{2}\right)(16 + x^2)^{-1/2} \cdot 2x + 2\left(\frac{1}{2}\right)(100 - 16x + x^2)^{-1/2}(2x - 16) = \frac{5x}{\sqrt{16+x^2}} + \frac{2x-16}{\sqrt{100-16x+x^2}} = 0$$

$x \approx 1.2529$

$C(0) = 40$

$C(1.2529) \approx 39.0162$

$C(8) \approx 56.7214$

The highway should emerge from the marsh 1.2529 miles east of the bridge.

(b) If we build a straight line to the interchange, we have $x = (3.2)$.

Since $C(3.2) - C(1.2529) \approx 1.963$, we save \$1.963 million.

22. (a) Say the road intersects the edge of the marsh at a distance x from the closest point on the edge to the bridge. Then x will be between 0 and 8. The length of road over marsh is now $M = \sqrt{x^2 + 4^2}$, and the length of road constructed on dry land will be $L = \sqrt{(8-x)^2 + 6^2}$. The total cost will be $C = 6M + 2L$.

We numerically solve
$C' = \frac{6x}{\sqrt{16 + x^2}} + \frac{2x - 16}{\sqrt{x^2 - 16x + 100}} = 0$
to find $x \approx 1.04345$. The first derivative test shows that this gives a minimum. The cost at this value is \$43.1763 million. The cost when we use the solution $x = 1.2529$ from exercise 19 is \$43.2078 million, so the increase is \$31,500.

(b) $C(x) = 5\sqrt{16 + x^2} + 3\sqrt{36 + (8-x)^2}$
$0 \le x \le 8$

$C'(x) = \frac{5x}{\sqrt{16+x^2}} + \frac{3x - 24}{\sqrt{100 - 16x + x^2}}$

Setting $C'(x) = 0$ yields

$x \approx 1.8941$

$C(0) = 50$

$C(1.8941) \approx 47.8104$

$C(8) \approx 62.7214$

The highway should emerge from the marsh 1.8941 miles east of the bridge. So if we must use the path from exercise 21, the extra cost is
$C(1.2529) - C(1.8941)$
$= 48.0452 - 47.8104 = 0.2348$
or about \$234.8 thousand.

23. Let r denote the running speed, and s be the swimming speed and let $f(y)$ be the time to get to the ball. Since time = distance/speed, we have

$$f(y) = \frac{z-y}{r} + \frac{\sqrt{x^2+y^2}}{s}$$

Solving $f'(y) = 0$ for y, we get $y = \dfrac{sx}{\sqrt{r^2-s^2}}$.

Substitute $x = 4$m and $z = 8$m. Therefore, we have

$$y = \frac{(0.9)\,(4)}{\sqrt{(6.4)^2 - (0.9)^2}}$$
$$\approx 0.56815.$$

Therefore, Elvis should enter into the water at $y \approx 0.56815$.

From the equation $y = \dfrac{x}{\sqrt{r/s+1}\sqrt{r/s-1}}$, we get that the value of y is independent of $z > 1$.

24. Let r denote the running speed, and s be the swimming speed and let $f(y)$ be the time to get to the ball. Since time = distance/speed, we have

$$f(y) = \frac{z-y}{r} + \frac{\sqrt{x^2+y^2}}{s}$$

Solving $f'(y) = 0$ for y, we get $y = \dfrac{sx}{\sqrt{r^2-s^2}}$.

Substitute $r = 6.4$ and $s = 0.9$.

$$y = \frac{0.9x}{\sqrt{(6.4)^2 - (0.9)^2}} \approx 0.144x$$

Therefore, for any x the optimal entry point is approximately $y = 0.144x$.

25. $T(x) = \dfrac{\sqrt{1+x^2}}{v_1} + \dfrac{\sqrt{1+(2-x)^2}}{v_2}$

$$T'(x) = \frac{1}{v_1}\cdot\frac{1}{2}(1+x^2)^{-1/2}\cdot 2x$$
$$+\frac{1}{v_2}(1+(2-x)^2)^{-1/2}\cdot(2-x)(-1)$$
$$= \frac{x}{v_1\sqrt{1+x^2}} + \frac{x-2}{v_2\sqrt{1+(2-x)^2}}$$

Note that
$$T'(x) = \frac{1}{v_1}\cdot\frac{x}{\sqrt{1+x^2}}$$
$$-\frac{1}{v_2}\cdot\frac{(2-x)}{\sqrt{1+(2-x)^2}}$$
$$= \frac{1}{v_1}\sin\theta_1 - \frac{1}{v_2}\sin\theta_2$$

When $T'(x) = 0$, we have
$$\frac{1}{v_1}\sin\theta_1 = \frac{1}{v_2}\sin\theta_2$$
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

26. The distance light travels is
$D = \sqrt{2^2+x^2} + \sqrt{1^2+(4-x)^2}$.
We maximize this by solving
$$D' = \frac{x}{\sqrt{4+x^2}} + \frac{2x-8}{2\sqrt{x^2-8x+17}} = 0$$
to find $x = 8/3$. For this value of x, $\theta_1 = \theta_2 = \tan^{-1}(3/4)$. (Or simply note similar triangles.)

27. $V(r) = cr^2(r_0 - r)$
$$V'(r) = 2cr(r_0-r) + cr^2(-1)$$
$$= 2crr_0 - 3cr^2$$
$$= cr(2r_0 - 3r)$$
$V'(r) = 0$ when $r = 2r_0/3$
$V'(r) > 0$ on $(0, 2r_0/3)$
$V'(r) < 0$ on $(2r_0/3, \infty)$
Thus $r = 2r_0/3$ maximizes the velocity.
$r = 2r_0/3 < r_0$, so the windpipe contracts.

28. We wish to minimize
$E(\theta) = \dfrac{\csc\theta}{r^4} + \dfrac{1-\cot\theta}{R^4}$.
We find

$$E'(\theta) = -\frac{\csc\theta\cot\theta}{r^4} + \frac{1+\cot^2\theta}{R^4}$$
$$= \frac{-\cos\theta R^4 + r^4}{r^4R^4\sin^2\theta}.$$

This is zero when $\cos\theta = r^4/R^4$, so $\theta = \cos^{-1}(r^4/R^4)$. The derivative changes from negative to positive here, so this gives a minimum as desired.

29. $p(x) = \dfrac{V^2 x}{(R+x)^2}$

$p'(x) = \dfrac{V^2(R+x)^2 - V^2 x \cdot 2(R+x)}{(R+x)^4}$

$= \dfrac{V^2R^2 - V^2x^2}{(R+x)^4}$

$p'(x) = 0$ when $x = R$

$p'(x) > 0$ on $(0, R)$

$p'(x) < 0$ on (R, ∞)

Thus $x = R$ maximizes the power absorbed.

30. If the meter registers 115 volts, then $v = 115\sqrt{2}$. The function $V(t) = v\sin(2\pi f t)$ has amplitude v, so the maximum value of the voltage is $115\sqrt{2}$.

31. $\pi r + 4r + 2w = 8 + \pi$

$w = \dfrac{8 + \pi - r(\pi + 4)}{2}$

$A(r) = \dfrac{\pi r^2}{2} + 2rw$

$= \dfrac{\pi r^2}{2} + r(8 + \pi - r(\pi + 4))$

$= r^2\left(-4 - \dfrac{\pi}{2}\right) + r(8 + \pi)$

$A'(r) = -2r\left(4 + \dfrac{\pi}{2}\right) + (8 + \pi) = 0$

$A'(r) = 0$ when $r = 1$

$A'(r) > 0$ on $(0, 1)$

$A'(r) < 0$ on $(1, \infty)$

Thus $r = 1$ maximizes the area so

$w = \dfrac{8 + \pi - (\pi + 4)}{2} = 2.$

The dimensions of the rectangle are 2×2.

32. Let x be the distance from the end at which the wire is cut. Due to symmetry, we may consider $0 \le x \le 1$. We wish to minimize the area of the squares formed by the two pieces. The total area is

$A(x) = \left(\dfrac{x}{4}\right)^2 + \left(\dfrac{2-x}{4}\right)^2$

$= \dfrac{2x^2 - 4x + 4}{16}.$

We compute

$A'(x) = \dfrac{x}{4} - \dfrac{1}{4} = 0$ when $x = 1$.

$A'' = \dfrac{1}{4} > 0$, so this is a minimum.

We check $A(0) = 1/4$ and $A(1) = 1/8$ and see that cutting the wire in half minimizes the area of the two squares.

33. $l \times w = 92$, $w = 92/l$

$A(l) = (l+4)(w+2)$

$= (l+4)(92/l + 2)$

$= 92 + 368/l + 2l + 8$

$= 100 + 368l^{-1} + 2l$

$A'(l) = -368l^{-2} + 2$

$= \dfrac{2l^2 - 368}{l^2}$

$A'(l) = 0$ when $l = \sqrt{184} = 2\sqrt{46}$

$A'(l) < 0$ on $(0, 2\sqrt{46})$

$A'(l) > 0$ on $(2\sqrt{46}, \infty)$

So $l = 2\sqrt{46}$ minimizes the total area. When $l = 2\sqrt{46}$, $w = \frac{92}{2\sqrt{46}} = \sqrt{46}$.

For the minimum total area, the printed area has width $\sqrt{46}$ in. and length $2\sqrt{46}$ in., and the advertisement has overall width $\sqrt{46}+2$ in. and overall length $2\sqrt{46} + 4$ in.

34. Let x and y be the width and height of the advertisement. Then $xy = 120$ and $y = 120/x$. We wish to maximize the printed area

$A = (x-2)(y-3) = (x-2)(\dfrac{120}{x} - 3)$

$= 126 - 3x - \dfrac{240}{x}.$

We find $A' = -3 + \dfrac{240}{x^2} = 0$ when $x = 4\sqrt{5}$. The first Derivative Test shows that this is a maximum. The smallest x could be is 2, and this gives $A(2) = 0$. The largest x could be is 40, and this also gives $A(40) = 0$. Thus, we see that the dimensions which maximize the printed area are $x = 4\sqrt{5}$ and $y = 6\sqrt{5}$.

35. (a) Let L represent the length of the ladder. Then from the diagram, it follows that $L = a\sec\theta + b\csc\theta$.

Therefore,

$$\frac{dL}{d\theta} = a\sec\theta\tan\theta - b\csc\theta\cot\theta$$

$$0 = a\sec\theta\tan\theta - b\csc\theta\cot\theta$$

$$a\sec\theta\tan\theta = b\csc\theta\cot\theta$$

$$\frac{b}{a} = \frac{\sec\theta\tan\theta}{\csc\theta\cot\theta} = \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}\frac{\sin\theta}{1}\frac{\sin\theta}{\cos\theta} = \tan^3\theta$$

Thus,

$\tan\theta = \sqrt[3]{b/a}$

$\theta = \tan^{-1}\left(\sqrt[3]{b/a}\right)$

$= \tan^{-1}\left(\sqrt[3]{4/5}\right)$

≈ 0.748 rad or 42.87 degrees

Thus, the length of the longest ladder that can fit around the corner is approximately

$L = a\sec\theta + b\csc\theta$

$= 5\sec(0.748) + 4\csc(0.748)$

≈ 12.7 ft

(b) From part (a), we have that $\theta = \tan^{-1}(\sqrt[3]{b/a})$ is the critical number limiting the length of the ladder. Thus $\tan\theta = b^{1/3}/a^{1/3}$. We can then draw a right triangle with θ as one angle and the length of the side opposite θ equal to $b^{1/3}$ and the length of the side adjacent to θ equal to $a^{1/3}$. By the Pythagorean Theorem, the hypotenuse of this triangle is $(a^{2/3}+b^{2/3})^{1/2}$. From this triangle, we find

$\sin\theta = \dfrac{b^{1/3}}{(a^{2/3}+b^{2/3})^{1/2}}$ and

$\cos\theta = \dfrac{a^{1/3}}{(a^{2/3}+b^{2/3})^{1/2}}$

so

$\csc\theta = \dfrac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}}$ and

$\sec\theta = \dfrac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}}$.

Thus

$$\begin{aligned} L &= a\sec\theta + b\csc\theta \\ &= a\cdot\frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}} + b\cdot\frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}} \\ &= a^{2/3}(a^{2/3}+b^{2/3})^{1/2} + b^{2/3}(a^{2/3}+b^{2/3})^{1/2} \\ &= (a^{2/3}+b^{2/3})(a^{2/3}+b^{2/3})^{1/2} \\ &= (a^{2/3}+b^{2/3})^{3/2}. \end{aligned}$$

(c) Using the result of part (b) and solving for b:

$$\begin{aligned} L &= (a^{2/3}+b^{2/3})^{3/2} \\ L^{2/3} &= a^{2/3}+b^{2/3} \\ b^{2/3} &= L^{2/3}-a^{2/3} \\ b &= (L^{2/3}-a^{2/3})^{3/2} \\ &= (8^{2/3}-5^{2/3})^{3/2} \\ &\approx 1.16 \text{ ft} \end{aligned}$$

(d) This was already done in part (c) while solving for b:
$b = (L^{2/3}-a^{2/3})^{3/2}$.

36. (a) $R(x) = \dfrac{35x - x^2}{x^2+35}$

$$\begin{aligned} R'(x) &= -35\frac{x^2+2x-35}{(x^2+35)^2} \\ &= -35\frac{(x-5)(x+7)}{(x^2+35)^2} \end{aligned}$$

Hence the only critical number for $x \ge 0$ is $x = 5$ (that is, 5000 items). This must correspond to the absolute maximum, since $R(0) = 0$ and $R(x)$ is negative for large x. So maximum revenue is $R(5) = 2.5$ (that is, \$2500).

(b) To maximize
$R(x) = \dfrac{cx - x^2}{x^2+c}$,
we compute
$R'(x) = \dfrac{c(c-2x-x^2)}{(x^2+c)^2}$.
This is zero when $x^2+2x-c=0$, so
$x = \dfrac{-2\pm\sqrt{4+4c}}{2}$.
The First Derivative Test shows that
$x = \dfrac{-2+\sqrt{4+4c}}{2}$
is a maximum.

37. (a) $Q'(t)$ is efficiency because it represents the number of additional items produced per unit time.

$$\begin{aligned} Q(t) &= -t^3+12t^2+60t \\ Q'(t) &= -3t^2+24t+60 \\ &= 3(-t^2+8t+20) \end{aligned}$$

This is the quantity we want to maximize. $Q''(t) = 3(-2t+8)$ so the only critical number is $t = 4$ hours. This must be the maximum since the function $Q'(t)$ is a parabola opening down.

(b) The worker's efficiency, Q' is maximized at the point of diminishing returns because at this point Q'' changes from positive to negative. The First Derivative Test applied to Q' shows that Q' has a local maximum at this point. (This assumes that the graph of Q changes from concave up to concave down at the inflection point. If this was reversed, the inflection point would not be a point of diminishing returns, and the efficiency would be minimized at such a point.)

38. (a) Let $C(t)$ be the total cost of the tickets. Then

$C(t) =$(price per ticket)(# of tickets)

$$\begin{aligned} C(t) &= (40-(t-20))(t) \\ &= (60-t)(t) = 60t - t^2 \end{aligned}$$

for $20 < t < 50$. Then $C'(t) = 60-2t$, so $t = 30$ is the only critical number. This must correspond to the maximum since $C(t)$ is a parabola opening down.

(b) If each additional ticket over 20 reduces the cost-per-ticket by c dollars, then the total cost for ordering x tickets (with x between 20 and 50) is

$$\begin{aligned} C(x) &= (40-c(x-20))x \\ &= (40+20c)x - cx^2. \end{aligned}$$

This is a downward facing parabola with one maximum at $x = \dfrac{20+10c}{c}$. If we want the maximum cost to be at $x = 50$, we must choose c so that the peak of the parabola is at or to the right of 50. The value of $x = \dfrac{20+10c}{c}$ increases as c decreases, and equals 50 when $c = \frac{1}{2}$. Any discount of 50 cents or less will cause the maximum cost to occur when the group orders 50 tickets.

39. $R = \dfrac{2v^2\cos^2\theta}{g}(\tan\theta - \tan\beta)$

$$\begin{aligned}
R'(\theta) &= \frac{2v^2}{g}\left[2\cos\theta(-\sin\theta)(\tan\theta-\tan\beta) + \cos^2\theta\cdot\sec^2\theta\right]\\
&= \frac{2v^2}{g}\left[-2\cos\theta\sin\theta\cdot\frac{\sin\theta}{\cos\theta} + 2\cos\theta\sin\theta\tan\beta + \cos^2\theta\cdot\frac{1}{\cos^2\theta}\right]\\
&= \frac{2v^2}{g}\left[-2\sin^2\theta+\sin(2\theta)\tan\beta+1\right]\\
&= \frac{2v^2}{g}\left[-2\sin^2\theta+\sin(2\theta)\tan\beta + (\sin^2\theta+\cos^2\theta)\right]\\
&= \frac{2v^2}{g}\left[\sin(2\theta)\tan\beta + (\cos^2\theta-\sin^2\theta)\right]\\
&= \frac{2v^2}{g}\left[\sin(2\theta)\tan\beta+\cos(2\theta)\right]
\end{aligned}$$

$R'(\theta) = 0$ when

$$\begin{aligned}
\tan\beta &= \frac{-\cos(2\theta)}{\sin(2\theta)} = -\cot(2\theta)\\
&= -\tan\left(\frac{\pi}{2}-2\theta\right)\\
&= \tan\left(2\theta-\frac{\pi}{2}\right)
\end{aligned}$$

Hence $\beta = 2\theta - \pi/2$, so

$$\begin{aligned}
\theta &= \frac{1}{2}\left(\beta+\frac{\pi}{2}\right)\\
&= \frac{\beta}{2}+\frac{\pi}{4} = \frac{\beta^\circ}{2}+45^\circ
\end{aligned}$$

i. $\beta = 10^\circ$, $\theta = 50^\circ$

ii. $\beta = 0^\circ$, $\theta = 45^\circ$

iii. $\beta = -10^\circ$, $\theta = 40^\circ$

40. $A = 4xy$

$$\frac{dA}{dx} = 4(xy'+y)$$

To determine $y' = \dfrac{dy}{dx}$, use the equation for the ellipse:

$$\begin{aligned}
1 &= \frac{x^2}{a^2}+\frac{y^2}{b^2}\\
0 &= \frac{2x}{a^2}+\frac{2yy'}{b^2}\\
\frac{2yy'}{b^2} &= -\frac{2x}{a^2}\\
y' &= -\frac{b^2}{a^2}\frac{x}{y}
\end{aligned}$$

Substituting this expression for y' into the expression for $\dfrac{dA}{dx}$, we get

$$\begin{aligned}
\frac{dA}{dx} &= xy'+y\\
&= x\left(-\frac{b^2}{a^2}\frac{x}{y}\right)+y\\
&= -\frac{b^2}{a^2}\frac{x^2}{y}+y
\end{aligned}$$

The area is maximized when its derivative is zero:

$$0 = -\frac{b^2}{a^2}\frac{x^2}{y}+y$$

$$\frac{b^2}{a^2}\frac{x^2}{y} = y$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

Substituting the previous relationship into the equation for the ellipse, we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{1}{2}$$

and therefore,

$$x = \frac{a}{\sqrt{2}} \quad\text{and}\quad y = \frac{b}{\sqrt{2}}$$

Thus, the maximum area is

$$A = 4\frac{a}{\sqrt{2}}\frac{b}{\sqrt{2}} = 2ab$$

Since the area of the circumscribed rectangle is $4ab$, the required ratio is

$$2ab : \pi ab : 4ab = 1 : \frac{\pi}{2} : 2$$

41. Let V_c be the volume of the cylinder, h be the height of the cylinder and r the radius of the cylinder so that

$V_c = h\pi r^2$.

Let V_s be the volume of the sphere and R be the radius of the sphere so that

$V_s = \dfrac{4}{3}\pi R^3$.

Draw the sphere on coordinate axes with center $(0,0)$ and inscribe the cylinder. Then draw a right triangle as follows: draw a straight line from the origin to the side of the cylinder (this line has length r, the radius of the cylinder); draw a line from this point to the point where the cylinder meets the sphere (this line has length $h/2$, half the height of the cylinder); connect this point with the origin to create the

hypotenuse of the triangle (this line has length R, the radius of the sphere). Thus we see that
$R^2 = r^2 + \left(\frac{h}{2}\right)^2$.
Now we have
$V_s = \frac{4}{3}\pi\left(r^2 + \frac{h^2}{4}\right)^{3/2}$.
Taking the derivative of both sides with respect to h gives
$0 = 2\pi\left(r^2 + \frac{h^2}{4}\right)^{1/2}\left(2rr' + \frac{h}{2}\right)$.
Solving for r', we find $r' = -h/4r$. Taking the derivative with respect to h of both sides of the formula for the volume for the cylinder yields
$\frac{dV_c}{dh} = \pi r^2 + 2h\pi rr'$.
Plugging in the formula we found for r' gives
$$\frac{dV_c}{dh} = \pi r^2 + 2h\pi r\left(\frac{-h}{4r}\right)$$
$$= \pi r^2 - \frac{h^2\pi}{2}.$$
To maximize the volume of the cylinder, we set this equal to 0 and find that the volume of the cylinder is maximized when $h^2 = 2r^2$. In this case, the formula relating R, r and h above gives
$h = \sqrt{\frac{4}{3}R^2} = \frac{2R}{\sqrt{3}}$.
The maximum volume of the cylinder is then
$$V_c = h\pi r^2$$
$$= \frac{\pi h^3}{2} = \frac{\pi\left(\frac{2R}{\sqrt{3}}\right)^3}{2}$$
$$= \frac{1}{\sqrt{3}}\left(\frac{4}{3}\pi R^3\right)$$
$$= \frac{1}{\sqrt{3}}V_s.$$

42. Suppose that $a = b$ in the isoscles triangle, so that
$A^2 = s(s-a)(s-b)(s-c) = s(s-a)^2(s-c)$
Since $s = \frac{1}{2}(a+b+c)$, it follows that $s = \frac{1}{2}(2a+c) = a + \frac{c}{2}$, so that $s - a = \frac{c}{2}$. Thus,
$$A^2 = s\left(\frac{c^2}{4}\right)(s-c)$$
$$= \frac{s}{4}\left(sc^2 - c^3\right)$$
Since s is a constant (it's half of the perimeter), we can now differentiate to get
$$2A\frac{dA}{dc} = \frac{s}{4}\left(2sc - 3c^2\right)$$
$$0 = c(2s - 3c)$$
Thus, the area is maximized when $2s - 3c = 0$, which means $c = \frac{2}{3}s$. Solving for a, we get
$$a = s - \frac{c}{2} = s - \frac{s}{3} = \frac{2}{3}s.$$
Thus, the area is maximized when $a = b = c$; in other words the area is maximized when the triangle is equilateral.

The maximum area is
$$A = \sqrt{s(s-c)^3} = \sqrt{s\left(\frac{s}{3}\right)^3}$$
$$= \frac{s^2}{9}\sqrt{3} = \frac{p^2}{36}\sqrt{3}$$

3.8 Related Rates

1. $V(t) = \text{(depth)(area)} = \frac{\pi}{48}\left[r(t)\right]^2$ (units in cubic feet per min)
$V'(t) = \frac{\pi}{48}2r(t)r'(t) = \frac{\pi}{24}r(t)r'(t)$
We are given $V'(t) = \frac{120}{7.5} = 16$.
Hence $16 = \frac{\pi}{24}r(t)r'(t)$ so
$r'(t) = \frac{(16)(24)}{\pi r(t)}$.

(a) When $r = 100$,
$r'(t) = \frac{(16)(24)}{100\pi} = \frac{96}{25\pi}$
≈ 1.2223 ft/min,

(b) When $r = 200$,
$r'(t) = \frac{(16)(24)}{200\pi} = \frac{48}{25\pi}$
≈ 0.61115 ft/min

2. $V = \text{(depth)(area)}$. $\frac{1}{8}'' = \frac{1}{96}'$, so
$V(t) = \frac{1}{96}\pi r(t)^2$.
Differentiating we find $\frac{dV}{dt} = \frac{2\pi}{96}r(t)\frac{dr}{dt}$.
Using 1 ft^3 = 7.5 gal, the rate of change of volume is $\frac{90}{7.5} = 12$. So when $r(t) = 100$,
$12 = \frac{2\pi}{96}100\frac{dr}{dt}$, and
$\frac{dr}{dt} = \frac{144}{25\pi}$ feet per minute.

3. (a) From #1,
$V'(t) = \frac{\pi}{48}2r(t)r'(t) = \frac{\pi}{24}r(t)r'(t)$,
so $\frac{g}{7.5} = \frac{\pi}{24}(100)(.6) = 2.5\pi$,
so $g = (7.5)(2.5)\pi$
$= 18.75\pi \approx 58.905$ gal/min.

(b) If the thickness is doubled, then the rate of change of the radius is halved.

4. (a) $t =$ hours elapsed since injury
$r =$ radius of the infected area
$A =$ area of the infection
$A = \pi r^2$
$A'(t) = 2\pi r(t) \cdot r'(t)$
When $r = 3$ mm, $r' = 1$ mm/hr,
$A' = 2\pi(3)(1) = 6\pi$ mm^2/hr

(b) We have $A'(t) = 2\pi r r'(t)$, and $r'(t) = 1$ mm/hr, so when the radius is 6 mm we have
$A'(t) = 2\pi \cdot 6 \cdot 1 = 12\pi \, \text{mm}^2/\text{hr}$.
This rate is larger when the radius is larger because the area is changing by the same amount along the entire circumference of the circle. When the radius is larger, there is more circumerence, so the same change in radius causes a larger change in area.

5. $V(t) = \dfrac{4}{3}\pi[r(t)]^3$
$V'(t) = 4\pi[r(t)]^2 r'(t) = Ar'(t)$
If $V'(t) = kA(t)$, then
$r'(t) = \dfrac{V'(t)}{A(t)} = \dfrac{kA(t)}{A(t)} = k.$

6. We have $A'(t) = 2\pi r r'(t)$, and $r'(t) = 5$ ft/min, so when the radius is 200 ft we have
$A'(t) = 2\pi \cdot 200 \cdot 5 = 2{,}000\pi$ ft^2/min.

7. (a) $10^2 = x^2 + y^2$

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$
$$= -\frac{6}{8}(3)$$
$$= -2.25 \text{ ft/s}$$

(b) We have
$\cos\theta(t) = \dfrac{x(t)}{10}$.
Differentiating with respect to t gives
$-\sin\theta(t) \cdot \theta'(t) = \dfrac{x'(t)}{10}$.
When the bottom is 6 feet from the wall, the top of the ladder is 8 feet from the floor and this distance is the opposite side of the triangle from *theta*. Thus, at this point, $\sin\theta = 8/10$. So
$-\dfrac{8}{10}\theta'(t) = \dfrac{3}{10}$
$\theta'(t) = -\dfrac{3}{8}$ rad/s.

8. (a) $\theta = \pi - \tan^{-1}\left(\dfrac{40}{60-x}\right) - \tan^{-1}\left(\dfrac{20}{x}\right)$

$$\frac{d\theta}{dx} = -\frac{40\left(\frac{1}{60-x}\right)^2}{1+\left(\frac{40}{60-x}\right)^2} + \frac{\frac{20}{x^2}}{1+\left(\frac{20}{x}\right)^2}$$

When $x = 30$, this becomes

$$\frac{d\theta}{dx} = -\frac{40\left(\frac{1}{30}\right)^2}{1+\left(\frac{40}{30}\right)^2} + \frac{\frac{20}{900}}{1+\left(\frac{20}{30}\right)^2}$$
$$= -\frac{1}{1625} \text{ rad/ft}$$
$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt}$$
$$= \left(-\frac{1}{1625}\right)(4)$$
$$\approx -0.00246 \text{ rad/s}$$

(b) As in the solution to #8(a), let x be the distance from the $20'$ building to the person. To find the maximum θ, we set $\dfrac{d\theta}{dx} = 0$ and solve for x:

$$0 = -\frac{40\left(\dfrac{1}{60-x}\right)^2}{1+\left(\dfrac{40}{60-x}\right)^2} + \frac{\dfrac{20}{x^2}}{1+\left(\dfrac{20}{x}\right)^2}$$
$$\frac{20}{x^2+40} = \frac{40}{(60-x)^2+1}$$

$0 = 20x^2 + 2400x - 56000$
$0 = x^2 + 120x - 2800$
Using the quadratic formula, we find two roots:
$x = -60 \pm 80$
We discard the x value obtained from the minus sign as it is negative and does not make sense for our problem. The other value is $x = 20$. We find $\theta'(10) > 0$ and $\theta'(30) < 0$, so $x = 20$ must be a maximum as desired.

9. (a) We know $[x(t)]^2 + 4^2 = [s(t)]^2$. Hence $2x(t)x'(t) = 2s(t)s'(t)$, so
$x'(t) = \dfrac{s(t)s'(t)}{x(t)} = \dfrac{-240s(t)}{x(t)}$. When $x = 40$, $s = \sqrt{40^2+4^2} = 4\sqrt{101}$, so at that moment
$x'(t) = \dfrac{(-240)(4\sqrt{101})}{40} = -24\sqrt{101}$.
So the speed is $24\sqrt{101} \approx 241.2$mph.

(b) From #9(a), we have
$x'(t) = \dfrac{s(t)s'(t)}{x(t)} = \dfrac{-240s(t)}{x(t)}$.
This time the height is 6 miles, so $s = \sqrt{40^2+6^2} = 2\sqrt{409}$, so at that moment
$x'(t) = \dfrac{(-240)(2\sqrt{409})}{40} = -12\sqrt{409}$.

So the speed is $12\sqrt{409} \approx 242.7$mph. The difference in height does not make a large difference in the speed of the plane.

10. (a) If the police car is not moving, then $x'(t) = 0$, but all the other data are unchanged. So

$$\begin{aligned} d'(t) &= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{[x(t)]^2 + [y(t)]^2}} \\ &= \frac{-(1/2)(50)}{\sqrt{1/4 + 1/16}} \\ &= \frac{-100}{\sqrt{5}} \approx -44.721. \end{aligned}$$

This is more accurate.

(b) If the police car is at the intersection, then the rate of change the police car measures is

$$\frac{0 \cdot (-40) + \frac{1}{2} \cdot (-50)}{\sqrt{\frac{1}{4} + 0}} = -50,$$

the true speed of the car.

11. $$\begin{aligned} d'(t) &= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{[x(t)]^2 + [y(t)]^2}} \\ &= \frac{-(1/2)(\sqrt{2}-1)(50) - (1/2)(50)}{\sqrt{1/4 + 1/4}} \\ &= -50. \end{aligned}$$

12. The radar gun will read less than the actual speed if the police car is not at the intersection, and is travelling away from the intersection.

13. From the table, we see that the recent trend is for advertising to increase by \$2000 per year. A good estimate is then $x'(2) \approx 2$ (in units of thousands). Starting with the sales equation
$s(t) = 60 - 40e^{-0.05x(t)}$,
we use the chain rule to obtain
$$\begin{aligned} s'(t) &= -40e^{-0.05x(t)}[-0.05x'(t)] \\ &= 2x'(t)e^{-0.05x(t)}. \end{aligned}$$
Using our estimate that $x'(2) \approx 2$ and since $x(2) = 20$, we get $s'(2) \approx 2(2)e^{-1} \approx 1.471$. Thus, sales are increasing at the rate of approximately \$1471 per year.

14. The year 2 rate of change for the average cost is given by $\overline{C}'(t) = \frac{-94}{x^2} \cdot x'(t)$.
From the table we see that in year two $x = 9.4$ and $x' = 0.6$, so
$\overline{C}'(t) = \frac{-94}{9.4^2} \cdot 0.6 = -0.6383$ per year.

15. $$\begin{aligned} \overline{C}(x) &= 10 + \frac{100}{x} \\ \overline{C}'(x(t)) &= \frac{-100}{x^2} \cdot x'(t) \end{aligned}$$

$\overline{C}'(10) = -1(2) = -2$ dollars per item, so average cost is decreasing at the rate of \$2 per year.

16. The rate of change of sales is
$s' = 0.8e^{-0.04x}x'(t)$.
We are given $x = 40$ and $x'(t) = 1.5$, so
$s' = 0.8e^{-0.04 \cdot 40} \cdot 1.5 = 0.242$ thousand dollars per year.

17. (a) We have $\tan\theta = \frac{x}{2}$, so

$$\begin{aligned} \frac{d}{dt}(\tan\theta) &= \frac{d}{dt}\left(\frac{x}{2}\right) \\ \sec^2\theta \cdot \theta' &= \frac{1}{2}x' \\ \theta' &= \frac{1}{2\sec^2\theta} \cdot x' = \frac{x'\cos^2\theta}{2} \end{aligned}$$

at $x = 0$, we have $\tan\theta = \frac{x}{2} = \frac{0}{2}$ so $\theta = 0$ and we have $x' = -130$ft/s so

$$\theta' = \frac{(-130) \cdot \cos^2 0}{2} = -65 \text{ rad/s}.$$

(b) $x = 2\tan\theta$, so $\frac{dx}{dt} = 2\sec^2\theta\frac{d\theta}{dt}$. $\theta = 0$ (and $\sec\theta = 1$) as the ball crosses home plate, so $\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$. For this to be less than 3 radians per sec, the pitch must be less than 6 ft/sec.

18. (a) t = number of seconds since launch
x = height of rocket in miles after t seconds
θ = camera angle in radians after t seconds

$$\begin{aligned} \tan\theta &= \frac{x}{2} \\ \frac{d}{dx}(\tan\theta) &= \frac{d}{dx}\left(\frac{x}{2}\right) \\ \sec^2\theta \cdot \theta' &= \frac{1}{2}x' \\ \theta' &= \frac{\cos^2\theta \cdot x'}{2} \end{aligned}$$

When $x = 3$, $\tan\theta = 3/2$, so $\cos\theta = 2/\sqrt{13}$.

$$\theta' = \frac{\left(\frac{2}{\sqrt{13}}\right)^2(.2)}{2} \approx .03 \text{ rad/s}$$

(b) If the height of the rocket is x, then $x = 2\tan\theta$, and
$\frac{dx}{dt} = 2\sec^2\theta\frac{d\theta}{dt}$.
When $x = 1$ and $\frac{dx}{dt} = 0.2$, we have $0.2 = 2 \cdot \frac{5}{4} \cdot \frac{d\theta}{dt}$ and $\frac{d\theta}{dt} = 0.08$ radians per sec. This is larger because the angle

changes more quickly when the rocket is close to the ground. When the rocket is far away, large changes in height result in small changes in the angle, since the angle is approaching a limit of $\pi/2$.

19. (a) Let θ be the angle between the end of the shadow and the top of the lamppost. Then $\tan\theta = \frac{6}{s}$ and $\tan\theta = \frac{18}{s+x}$, so

$$\frac{x+s}{18} = \frac{s}{6}$$
$$\frac{d}{dx}\left(\frac{x+s}{18}\right) = \frac{d}{dx}\left(\frac{s}{6}\right)$$
$$\frac{x'+s'}{18} = \frac{s'}{6}$$
$$x'+s' = 3s'$$
$$s' = \frac{x'}{2}$$

Since $x' = 2$, $s' = 2/2 = 1$ ft/s.

(b) From #19(a), $s' = x'/2$. Since $x' = -3$, $s' = -3/2$ ft/s.

20. (a) $P(t)\cdot V'(t) + P'(t)V(t) = 0$
$$\frac{P'(t)}{V'(t)} = -\frac{P(t)}{V(t)} = -\frac{c}{V(t)^2}$$

(b) Solving Boyle's Law for P gives $P = \frac{c}{V}$. Then differentiating gives $P'(V) = \frac{-c}{V^2}$, the same as $P'(t)/V'(t)$.

21. Let $r(t)$ be the length of the rope at time t and $x(t)$ be the distance (along the water) between the boat and the dock.

$$r(t)^2 = 36 + x(t)^2$$
$$2r(t)r'(t) = 2x(t)x'(t)$$
$$x'(t) = \frac{r(t)r'(t)}{x(t)} = \frac{-2r(t)}{x(t)}$$
$$= \frac{-2\sqrt{36+x^2}}{x}$$

When $x = 20$, $x' = -2.088$; when $x = 10$, $x' = -2.332$.

22. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, and we know that this cone has $r = \frac{h}{2}$, so we have $V = \frac{\pi}{12}h^3$. Differentiating gives

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\cdot\frac{dh}{dt}.$$

We are given that $\frac{dV}{dt} = 5\,\mathrm{m}^3/\mathrm{s}$, so when $h = 2$ meters, we have

$$5 = \frac{\pi 2^2}{4}\cdot\frac{dh}{dt},$$

so $\frac{dh}{dt} = \frac{5}{\pi}$ meters per second.

23. $f(t) = \frac{1}{2L(t)}\sqrt{\frac{T}{\rho}} = \frac{110}{L(t)}$.

$f'(t) = \frac{-110}{L(t)^2}L'(t)$.

When $L = 1/2$, $f(t) = 220$ cycles per second. If $L' = -4$ at this time, then $f'(t) = 1760$ cycles per second per second. It will only take 1/8 second at this rate for the frequency to go from 220 to 440, and raise the pitch one octave.

24. $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$
$$1 = 4\pi r^2\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{4\pi r^2}$$

When $r = .01$, $\frac{dr}{dt} = \frac{2500}{\pi}$

When $r = .1$, $\frac{dr}{dt} = \frac{25}{\pi}$.

At first, the radius expands rapidly; later it expands more slowly.

25. (a) Let R represent the radius of the circular surface of the water in the tank.

$$V(R) = \pi\left[60^2(60^2-R^2)^{1/2} - \frac{1}{3}(60^2-R^2)^{3/2} + \frac{2}{3}60^3\right]$$

$$\frac{dV}{dR} = \pi\left[60^2\left(\frac{1}{2}\right)(60^2-R^2)^{-1/2}(-2R) - \frac{1}{3}\left(\frac{3}{2}\right)(60^2-R^2)^{1/2}(-2R)\right]$$
$$= \pi\left[\frac{-60^2 R}{\sqrt{60^2-R^2}} + R\sqrt{60^2-R^2}\right]$$
$$= \pi R\left[\frac{-60^2+60^2-R^2}{\sqrt{60^2-R^2}}\right]$$
$$= \frac{-\pi R^3}{\sqrt{60^2-R^2}}$$
$$\frac{dR}{dt} = \frac{dV/dt}{dV/dR}$$
$$= \frac{10}{dV/dR}$$
$$= \frac{-10\sqrt{60^2-R^2}}{\pi R^3}$$

i. Substituting $R = 60$ into the previous equation, we get $\frac{dR}{dt} = 0$.

ii. We need to determine the value of R when the tank is three-quarters full. The volume of the spherical tank is $\frac{4}{3}\pi 60^3$, so when the tank is three-quarters full, $V(R) = \pi 60^3$. Substituting this value into the formula for $V(R)$ and solving for R (using a CAS, for example) we get $R \approx 56.265$. Substituting this value into the formula for dR/dt, we get

$$\begin{aligned}\frac{dR}{dt} &= \frac{-10\sqrt{60^2 - R^2}}{\pi R^3}\\ &\approx \frac{-10\sqrt{60^2 - 56.265^2}}{\pi 56.265^3}\\ &\approx -0.00037 \text{ ft/s}\end{aligned}$$

(b) Assuming the tank is at least half full, we can represent the height of the water in the tank by $h(t) = \sqrt{60^2 - R^2} + 60$. Differentiating gives

$$\begin{aligned}h'(t) &= \frac{1}{2}(60^2 - R^2)^{-1/2}(-2R)R'(t)\\ &= -(60^2 - R^2)^{-1/2}R \cdot R'(t)\\ &= \frac{-(60^2 - R^2)^{-1/2}R \cdot (-10\sqrt{60^2 - R^2})}{\pi R^3}.\end{aligned}$$

Here we have used the expression for $R'(t)$ found in exercise 35.

i. Substituting $R = 60$ into the previous equation, we get $h'(t) = 0$.

ii. Substituting $R \approx 56.265$ into the formula for $h'(t)$ gives $h'(t) \approx 0.001006$ ft/s.

26. (a) The volume of the conical pile is $V = \frac{1}{3}\pi r^2 h$. Since $h = 2r$, we can write the volume as

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$ Thus,

$$\begin{aligned}\frac{dV}{dt} &= \frac{\pi h^2}{4}\cdot\frac{dh}{dt}\\ 20 &= \frac{\pi 6^2}{4}\cdot\frac{dh}{dt}\\ \frac{dh}{dt} &= \frac{20}{9\pi}\\ \frac{dr}{dt} &= \frac{10}{9\pi}\end{aligned}$$

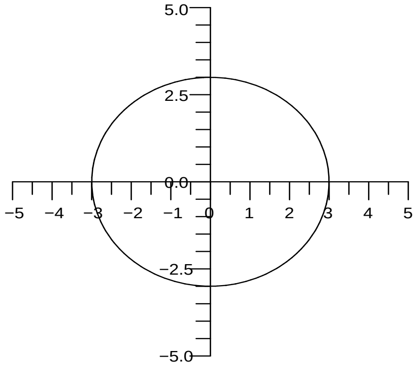
(b) In this case, we have $r = h$ so

$V = \frac{1}{3}\pi h^2 h = \frac{\pi h^3}{3}$

Thus $V'(t) = \pi h^2 h'(t)$ so when the height is 6 feet,

$h'(t) = r'(t) = \frac{20}{36\pi} = \frac{5}{9\pi}$.

27. (a) Let an object move around the circle $x^2(t) + y^2(t) = r^2$. Both x and y coordinates are the functions of t and r is a constant.



Therefore, on differentiating w.r.t. t, we get

$$\begin{aligned}&2x(t)x'(t) + 2y(t)y'(t) = 0\\ &x(t)x'(t) + y(t)y'(t) = 0\end{aligned}$$

Therefore, $y'(t) = \frac{-x(t)x'(t)}{y(t)}$ and

$x'(t) = -\frac{y(t)y'(t)}{x(t)}$

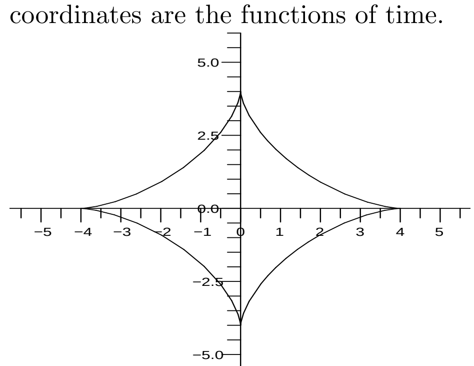
Thus, if $x(t) = 0$, then $y'(t) = 0$ and if $y(t) = 0$, then $x'(t) = 0$

From the graph it can be observed that:

At $x(t) = 0$ the tangent is horizontal which means $\frac{y'(t)}{x'(t)} = 0 \Rightarrow y'(t) = 0$ and

At $y(t) = 0$ the tangent is vertical which means $\frac{x'(t)}{y'(t)} = 0 \Rightarrow x'(t) = 0$

(b) An object move around the asteroid $x^{2/3}(t) + y^{2/3}(t) = 1$. Both x and y coordinates are the functions of time.

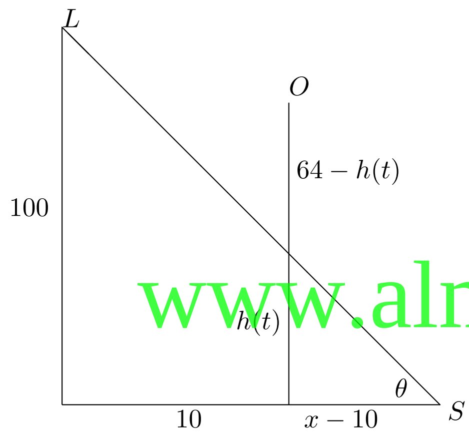


Therefore, on differentiating w.r.t. t, we get

$$\begin{aligned}&\frac{2}{3}x^{-1/3}(t)x'(t) + \frac{2}{3}y^{-1/3}(t)y'(t) = 0\\ &x(t)[y'(t)]^3 + y(t)[x'(t)]^3 = 0\end{aligned}$$

$$y'(t) = \left(\frac{-y(t)}{x(t)}\right)^{1/3} x'(t) \text{ and}$$
$$x'(t) = \left(\frac{-x(t)}{y(t)}\right)^{1/3} y'(t)$$
thus, if $x(t) = 0$, then $y(t) = 1$, $x'(t) = 0$ and if $y(t) = 0$, then $x(t) = 1$, $y'(t) = 0$
From the graph it can be observed that, at $x(t) = 0$ the tangent is vertical which means $\frac{x'(t)}{y'(t)} = 0 \Rightarrow x'(t) = 0$ and at $y(t) = 0$ the tangent is horizontal which means $\frac{y'(t)}{x'(t)} = 0 \Rightarrow y'(t) = 0$

28. (a) Let θ be the angle of the light at the shadow as shown in figure below:



Then,
$$\tan\theta = \frac{h(t)}{x(t)-10} = \frac{100}{x(t)}$$
$$\Rightarrow x(t) = \frac{1000}{100-h(t)}$$
$$x'(t) = \frac{1000h'(t)}{[100-h(t)]^2}$$
$$= -\frac{8000\sqrt{(64-h(t))}}{[100-h(t)]^2}$$
At $h = 0$,
$$x'(t) = -\frac{8000\sqrt{(64-0)}}{[100-0]^2}$$
$$= -\frac{64000}{10000}$$
$$= -6.4$$

(b) $|x'(t)| = 8000\frac{\sqrt{(64-h(t))}}{[100-h(t)]^2}$
At maxima or minima of $|x'(t)|$,
$$\frac{d}{dx}|x'(t)| = 0$$
$$\Rightarrow \frac{-\frac{1}{2}(64-h(t))^{-1/2}h'(t)}{[100-h(t)]^2}$$
$$+ \frac{\sqrt{(64-h(t)}}{[100-h(t)]^3} \cdot 2h'(t) = 0$$
$$\Rightarrow -h'(t)\{(100-h(t)) - 4(64-h(t))\} = 0$$
$$\Rightarrow h'(t) = 0 \text{ or } 100-256-h(t)+4h(t) = 0$$
$$\Rightarrow h'(t) = 0 \text{ or } h(t) = 52$$
At $h'(t) = 0 : |x'(t)| = 0$
At $h(t) = 52$:
$$|x'(t)| = 8000\frac{\sqrt{64-h(t)}}{[100-h(t)]^2}$$
$$= 8000\frac{\sqrt{(64-52)}}{(100-52)^2} = 12.02$$
Therefore, $h(t) = 52$ is the height in which $|x'(t)|$ is maximum.

29. (a) $d(t) = \sqrt{(x(t)-8)^2 + (0-4)^2}$
therefore $d'(t) = \frac{(x(t)-8)x'(t)}{\sqrt{(x(t)-8)^2+16}}$
Now $d'(t) = 0.9$ and $x'(t) = 6.4$
gives $x(t) = 8.5681$
hence $t = \frac{x(t)-x(0)}{x'(t)} = \frac{8.5681}{6.4} = 1.3388$

(b) Thus the location at this moment is $(8.5681,\ 0)$

30. $\theta = \tan^{-1}\left(\frac{2s}{vT}\right)$
$$\frac{d\theta}{dt} = \frac{\left(-\frac{2s}{T}\right)v^{-2}v'(t)}{1+\left(\frac{2s}{vT}\right)^2}$$
$$= \frac{-2sv'(t)}{Tv^2\left[1+\frac{4s^2}{v^2T^2}\right]}$$
$$= \frac{-2sTv'(t)}{T^2v^2+4s^2}$$
For $T = 1$, $s = 0.6$ and $v'(t) = 1$,
$$\frac{d\theta}{dT} = \frac{-1.2}{v^2+1.44}$$

(a)
$$\frac{d\theta}{dT} = \frac{-1.2}{2.44} \approx -0.4918 \text{ rad/s}$$

(b)
$$\frac{d\theta}{dT} = \frac{-1.2}{5.44} \approx -0.2206 \text{ rad/s}$$

3.9 Rates of Change in Economics and the Sciences

1. The marginal cost function is $C'(x) = 3x^2 + 40x + 90$.
The marginal cost at $x = 50$ is $C'(50) = 9590$. The cost of producing the 50th item is $C(50) - C(49) = 9421$.

2. The marginal cost function is
$C'(x) = 4x^3 + 28x + 60$.
The marginal cost at $x = 50$ is $C'(50) = 501460$. The cost of producing the 50th item is $C(50) - C(49) = 486645$.

3. The marginal cost function is
$C'(x) = 3x^2 + 42x + 110$.
The marginal cost at $x = 100$ is $C'(100) = 34310$. The cost of producing the 100th item is $C(100) - C(99) = 33990$.

4. The marginal cost function is
$C'(x) = 3x^2 + 22x + 40$.
The marginal cost at $x = 100$ is $C'(100) = 32240$. The cost of producing the 100th item is $C(100) - C(99) = 31930$.

5. $C'(x) = 3x^2 - 60x + 300$
$C''(x) = 6x - 60 = 0$
$x = 10$ is the inflection point because $C''(x)$ changes from negative to positive at this value. After this point, cost rises more sharply.

6. A linear model doesn't reflect the capacity of the stadium, or the presence of a certain number of fans who would attend no matter what the price, but away from the extremes a linear model might serve adequately. For ticket price x, the revenue function is
$R(x) = x(-3{,}000x + 57{,}000)$
$\quad = -3{,}000x^2 + 57{,}000x$.
We solve
$R'(x) = -6{,}000x + 57{,}000 = 0$
and find that $x = 9.5$ dollars per ticket is the critical number. Since $R'' = -6{,}000 < 0$, this is a maximum.

7. $\overline{C}(x) = C(x)/x = 0.1x + 3 + \dfrac{2000}{x}$
$\overline{C}'(x) = 0.1 - \dfrac{2000}{x^2}$
Critical number is $x = 100\sqrt{2} \approx 141.4$.
$\overline{C}'(x)$ is negative to the left of the critical number and positive to the right, so this must be the minimum.

8. The average cost function is
$$\overline{C}(x) = \frac{0.2x^3 + 4x + 4000}{x} = 0.2x^2 + 4 + \frac{4000}{x}.$$
$$\overline{C}'(x) = 0.4x - \frac{4000}{x^2} = 0$$
when $x \approx 21.54$. This is a minimum because $\overline{C}'' = 0.4 + \dfrac{4000}{x^3} > 0$ at this x.

9. $\overline{C}(x) = C(x)/x = 10\dfrac{e^{0.02x}}{x}$
$\overline{C}'(x) = 10e^{.02x}\left(\dfrac{.02x - 1}{x^2}\right)$
Critical number is $x = 50$. $\overline{C}'(x)$ is negative to the left of the critical number and positive to the right, so this must be the minimum.

10. The average cost function is
$\overline{C}(x) = \dfrac{\sqrt{x^3 + 800}}{x}$ and
$\overline{C}'(x) = \dfrac{x^3 - 1600}{2x^2\sqrt{x^3 + 800}}$.
This is zero when $x = \sqrt[3]{1600}$. This is a minimum because
$\overline{C}'' = \dfrac{5{,}120{,}000 + 12{,}800x^3 - x^6}{4x^3(x^3 + 800)^{3/2}} > 0$ at this x.

11. (a) $C(x) = 0.01x^2 + 40x + 3600$
$C'(x) = 0.02x + 40$
$\overline{C}(x) = \dfrac{C(x)}{x} = 0.01x + 40 + \dfrac{3600}{x}$
$C'(100) = 42$
$\overline{C}(100) = 77$
so $C'(100) < \overline{C}(100)$
$\overline{C}(101) = 76.65 < \overline{C}(100)$

(b) $C'(x) = 0.02x + 40$
$C'(1000) = 60$
$\overline{C}(x) = \dfrac{0.01x^2 + 40x + 3600}{x}$
$\overline{C}(1000) = 53.6$
$\overline{C}(1001) = 53.6064$

(c) $\overline{C}'(x) = 0.01 - \dfrac{3600}{x^2} = 0$
so $x = 600$ is min and
$C'(600) = 52$
$\overline{C}(600) = 52$

12. (a) $P(x) = R(x) - C(x)$
$P'(x) = R'(x) - C'(x) = 0$
$R'(x) = C'(x)$

(b) $P(x) = (10x - 0.001x^2) - (2x + 5{,}000)$.
$P'(x) = 8 - 0.002x = 0$ if $x = 4{,}000$.
This is a maximum because $P''(x) = -0.002 < 0$.

13. $E = \dfrac{p}{f(p)}f'(p)$
$\quad = \dfrac{p}{200(30 - p)}(-200) = \dfrac{p}{p - 30}$
To solve $\dfrac{p}{p - 30} < -1$, multiply both sides

by the negative quantity $p-30$, to get $p > (-1)(p-30)$ or $p > 30-p$, so $2p > 30$, so $15 < p < 30$.

14. $E = \dfrac{pf'(p)}{f(p)} = \dfrac{p(-200)}{200(20-p)} = \dfrac{p}{p-20}$

$\dfrac{p}{p-20} < -1$ when $p > 20-p$, so demand is elastic when $10 < p < 20$.

15. $f(p) = 100p(20-p) = 100(20p-p^2)$

$$\begin{aligned} E &= \frac{p}{f(p)} f'(p) \\ &= \frac{p}{100p(20-p)}(100)(20-2p) \\ &= \frac{20-2p}{20-p} \end{aligned}$$

To solve $\dfrac{20-2p}{20-p} < -1$, multiply both sides by the positive quantity $20-p$ to get $20-2p < (-1)(20-p)$, or $20-2p < p-20$, so $40 < 3p$, so $40/3 < p < 20$.

16. $$\begin{aligned} E &= \frac{pf'(p)}{f(p)} \\ &= \frac{p(600-120p)}{60p(10-p)} = \frac{2p-10}{p-10} \end{aligned}$$

If $\dfrac{2p-10}{p-10} < -1$ for positive p, then $p-10$ must be negative. this means $\dfrac{2p-10}{p-10} < -1$ when $2p-10 > 10-p$, so demand is elastic when $\dfrac{20}{3} < p < 10$.

17. $[pf(p)]' < 0$
if and only if $p'f(p) + pf'(p) < 0$
if and only if $f(p) + pf'(p) < 0$
if and only if $pf'(p) < -f(p)$
if and only if $\frac{pf'(p)}{f(p)} < -1$

18. The percentage change in quantity purchased (using the chain rule) is $\frac{Q'(I)\cdot I'}{Q(I)}$. The percentage change in income is $\frac{I'}{I}$.
The income elasticity of demand is then $\frac{Q'(I)\cdot I'}{Q(I)} \cdot \frac{I}{I'}$ or $\frac{Q'(I)\cdot I}{Q(I)}$.

19. (a) Rewrite $x'(t)$ as $f(x) = 2x[4-x]$.
$$\begin{aligned} f'(x) &= 2(4-x) + 2x(-1) \\ &= 8-4x \end{aligned}$$
$f'(x) = 0 \Rightarrow x = 2$ where the $f(x)$ is maximum

(b) The critical points of $x'(t) = 2x[4-x]$ are $x = 0$ and $x = 4$.
$x'(t) > 0,\ 0 < x(t) < 4$
$x'(t) < 0,\ x > 4$ or $x < 0$
Therefore, the limiting concentration is 4.

20. (a) Rewrite $x'(t)$ as $f(x) = 0.5x[5-x]$.
$$\begin{aligned} f'(x) &= 0.5(5-x) + 0.5x(-1) \\ &= 2.5-x \end{aligned}$$
$f'(x) = 0 \Rightarrow x = 2.5$ where the $f(x)$ is maximum.

(b) The critical points of $x'(t) = 0.5x[5-x]$ are $x = 0$ and $x = 5$.
$x'(t) > 0,\ 0 < x(t) < 5$
$x'(t) < 0,\ x > 5$ or $x < 0$
Therefore, the limiting concentration is 5.

21. $y'(t) = c \cdot y(t)[K - y(t)]$
$y(t) = Kx(t)$
$y'(t) = Kx'(t)$
$Kx'(t) = c \cdot Kx(t)[K - Kx(t)]$
$$\begin{aligned} x'(t) &= c \cdot Kx(t)[1-x(t)] \\ &= rx(t)[1-x(t)] \end{aligned}$$
$r = cK$

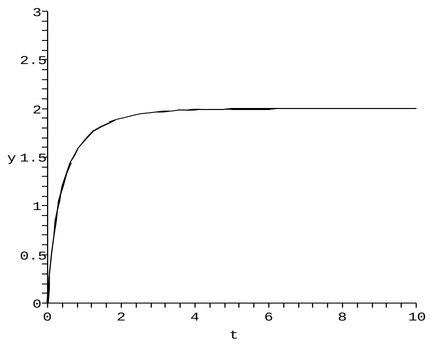
22. The given conditions translate into equations $3 = c \cdot 2(K-2)$ and $4 = c \cdot 4(K-4)$. Solving the first equation for c and substituting into the second equation gives
$4 = \frac{4\cdot 3(K-4)}{2(K-2)} \Rightarrow K = 8$ and $c = 1/4$.

23. $x'(t) = [a - x(t)][b - x(t)]$
for $x(t) = a$,
$x'(t) = [a-a][b-a] = 0$
So the concentration of product is staying the same.
If $a < b$ and $x(0) = 0$ then $x'(t) > 0$ for $0 < x < a < b$
$x'(t) < 0$ for $a < x < b$
Thus $x(t) = a$ is a maximum.

24. $$\begin{aligned} x(0) &= \frac{a[1-e^{-(b-a)\cdot 0}]}{1-\left(\frac{a}{b}\right)e^{-(b-a)\cdot 0}} \\ &= \frac{a[1-1]}{1-\left(\frac{a}{b}\right)} = 0 \end{aligned}$$

$\lim\limits_{t\to\infty} x(t) = \frac{a[1-0]}{1-0} = a$

For $a = 2$ and $b = 3$ the graph looks like this:



25. The first inflection point occurs around $f = 1/3$, before the step up. The second occurs at the far right of the graph. The equivalence point is presumably more stable. The first inflection point would be hard to measure, since the pH takes drastic leap right after the inflection point occurs.

26. Recall that we are assuming $0 < f < 1$. As $f \to 1^-$,
$p'(f) = \frac{1}{f(1-f)} \to \infty$

27. $R(x) = \dfrac{rx}{k+x}$, $x \geq 0$
$R'(x) = \dfrac{rk}{(k+x)^2}$
There are no critical numbers. Any possible maximum would have to be at the endpoint $x = 0$, but in fact R is increasing on $[0, \infty)$, so there is no maximum (although as x goes to infinity, R approaches r).

28. $PV^{7/5} = c$
$$\frac{d}{dP}\left(PV^{7/5}\right) = \frac{d}{dP}(c) = 0$$
$$V^{7/5} + \frac{7}{5}PV^{2/5}\frac{dV}{dP} = 0$$
$$V + \frac{7}{5}P\frac{dV}{dP} = 0$$
$$\frac{dV}{dP} = \frac{-5}{7}\frac{V}{P}.$$
But $V^{7/5} = c/P$, so $V = (c/P)^{5/7}$. Hence
$$\frac{dV}{dP} = \frac{-5}{7}\frac{V}{P} = \frac{-5}{7}\frac{(c/P)^{5/7}}{P} = \frac{-5c^{5/7}}{7P^{12/7}}.$$
As pressure increases, volume decreases.

29. $m'(x) = 4 - \cos x$, so the rod is less dense at the ends.

30. $m'(x) = 3(x-1)^2 + 6$.
Density is maximum at the ends and at a minimum in the middle.

31. $m'(x) = 4$, so the rod is homogeneous.

32. $m'(x) = 8x$.
Density increases from 0 at the left end to a maximum at the right end.

33.
$$\begin{aligned} Q'(t) &= e^{-2t} \cdot (-2)(\cos 3t - 2\sin 3t) \\ &\quad + e^{-2t}((-\sin 3t \cdot 3) - 2\cos 3t \cdot 3) \\ &= e^{-2t}(-8\cos 3t + \sin 3t) \text{ amps} \end{aligned}$$

34.
$$\begin{aligned} Q'(t) &= e^t(3\cos 2t + \sin 2t) \\ &\quad + e^t(-6\sin 2t + 2\cos 2t) \\ &= 5e^t(\cos 2t - \sin 2t) \text{ amps} \end{aligned}$$

35. As $t \to \infty$, $Q(t) \to 4\sin 3t$, so $e^{-3t}\cos 2t$ is called the transient term and $4\sin 3t$ is called the steady-state value.
$$\begin{aligned} Q'(t) &= e^{-3t} \cdot (-3)\cos 2t \\ &\quad + e^{-3t}(-\sin 2t \cdot 2) + 4\cos 3t \cdot 3 \\ &= e^{-3t}(-3\cos 2t - 2\sin 2t) \\ &\quad + 12\cos 3t \end{aligned}$$
The transient term is $e^{-3t}(-3\cos 2t - 2\sin 2t)$ and the steady-state value is $12\cos 3t$.

36.
$$\begin{aligned} Q'(t) &= -2e^{-2t}(\cos t - 2\sin t) \\ &\quad + e^{-2t}(-\sin t - 2\cos t) \\ &\quad + e^{-3t} - 3te^{-3t} - 8\sin 4t \end{aligned}$$

$$\begin{aligned} Q'(t) &= e^{-2t}(-4\cos t + 3\sin t) \\ &\quad + e^{-3t}(1-3t) - 8\sin 4t \end{aligned}$$
The transient term is $e^{-2t}(-4\cos t + 3\sin t) + e^{-3t}(1-3t)$ and the steady-state value is $-8\sin 4t$.

37. The rate of population growth is given by
$f(p) = 4p(5-p) = 4(5p - p^2)$
$f'(p) = 4(5 - 2p)$,
so the only critical number is $p = 2.5$. Since the graph of f is a parabola opening down, this must be a max.

38. The rate of growth $R = 2p(7-2p)$, so $R' = 14 - 8p = 0$ when $p = 7/4$. This is a maximum because $R'' = -8 < 0$.

39.
$$\begin{aligned} p'(t) &= \frac{-B(1+Ae^{-kt})'}{(1+Ae^{-kt})^2} \\ &= \frac{-B(-kAe^{-kt})}{(1+Ae^{-kt})^2} \\ &= \frac{kABe^{-kt}}{(1+Ae^{-kt})^2} \\ &= \frac{kABe^{-kt}}{1 + 2Ae^{-kt} + A^2e^{-2kt}} \\ &= \frac{kAB}{e^{kt} + 2A + A^2e^{-kt}} \end{aligned}$$
As t goes to infinity, the exponential term goes

to 0, and so the limiting population is
$\frac{B}{1+A(0)} = B.$

40. If the inflection point is $p = 120$, then the maximum population is $B = 240$. If the initial population is $p(0) = 40$, then
$40 = \frac{240}{1+A}.$
We solve to get $A = 5$. If then $p(12) = 160$, we have the equation
$160 = \frac{240}{1+5e^{-12k}}$
which we can solve to get
$k = \frac{\ln 10}{12}.$

41. For $a = 70$, $b = 0.2$, $f(t) = \frac{70}{1+3e^{-0.2t}} = 70(1+3e^{-0.2t})^{-1}$
$f(2) = \frac{7-0}{1+3e^{-0.2\cdot 2}} \approx 23$
$f'(t) = -70(1+3e^{-0.2t})^{-2}(3e^{-0.2t})(-0.2)$
$= \frac{42e^{-0.2t}}{(1+3e^{-0.2t})^2}$
$f'(2) = \frac{42e^{0.2\cdot 2}}{(1+3e^{-0.2\cdot 2})^2} \approx 3.105$

This says that at time $t = 2$ hours, the rate at which the spread of the rumor is increasing is about 3% of the population per hour.
$\lim_{t\to\infty} f(t) = \frac{70}{1+0} = 70$
so 70% of the population will eventually hear the rumor.

42. $f'(t) = -0.02e^{-0.02t} + 0.42e^{-0.42t}$
$f'(t) = 0$ when $0.42e^{-0.42t} = 0.02e^{-0.02t}$, or $e^{-0.4t} = 0.02/0.42$. So we see that
$t = -\frac{\ln 0.047619}{0.4} \approx 7.6113$
is the critical value. The Second Derivative Test shows that it is a maximum.

43. $f'(x) = \frac{-64x^{-1.4}(4x^{-0.4}+15)}{(4x^{-0.4}+15)^2}$
$- \frac{(160x^{-0.4}+90)(-1.6x^{-1.4})}{(4x^{-0.4}+15)^2}$
$= \frac{-816x^{-1.4}}{(4x^{-0.4}+15)^2} < 0$
So $f(x)$ is decreasing. This shows that pupils shrink as light increases.

44. $T(x) = 102 - \frac{1}{6}x^2 + \frac{1}{54}x^3.$
To maximize $|T'(x)|$, we find all extrema of $T'(x)$ and compare their magnitudes.
$T'(x) = \frac{-1}{3}x + \frac{1}{18}x^2.$
$T''(x) = \frac{-1}{3} + \frac{1}{9}x = 0$ when $x = 3$.
We test the critical numbers and the endpoints: $T'(0) = 0$, $T'(6) = 0$, and $T'(3) = \frac{-1}{2}$. The dosage that maximizes sensitivity is 3 mg.

45. If v is not greater than c, the fish will never make any headway. $E'(v) = \frac{v(v-2c)}{(v-c)^2}$ so the only critical number is $v = 2c$. When v is large, $E(v)$ is large, and when v is just a little bigger than c, $E(v)$ is large, so we must have a minimum.

46. We wish to minimize $P = \frac{1}{v} + cv^3$.
$P' = \frac{-1}{v^2} + 3cv^2 = 0$ when $v = \sqrt[4]{\frac{1}{3c}}$.
$P'' = \frac{2}{v^3} + 6cv > 0$ at this velocity, so this gives the minimum power.

47. (a) $xy = c$
$y = \frac{c}{x}$
Time spent to cover y miles $= \frac{y}{r_1}$
Time spent to cover x miles$= \frac{x}{r_2}$
So, the total time spent $(T) = \frac{y}{r_1} + \frac{x}{r_2}$
Now by taking $f(x) = T$ we get:
$f(x) = \left(\frac{y}{r_1} + \frac{x}{r_2}\right)$
$= \left(\frac{c}{r_1}\frac{1}{x} + \frac{x}{r_2}\right)$
$f'(x) = \frac{-c}{r_1}\cdot\frac{1}{x^2} + \frac{1}{r_2}$
$f'(x) = 0 \Rightarrow$
$-cr_2 + r_1x^2 = 0$
$r_1x^2 = cr_2$
$x^2 = \frac{cr_2}{r_1}$
$x = \sqrt{\frac{cr_2}{r_1}}$
Substitute $x = \sqrt{\frac{cr_2}{r_1}}$ in $y = \frac{c}{x}$.
$y = \frac{c}{\sqrt{\frac{cr_2}{r_1}}}$
$= \sqrt{\frac{r_1c}{r_2}}$
Therefore, when $x = \sqrt{\frac{cr_2}{r_1}}$ and $y = \sqrt{\frac{r_1c}{r_2}}$, the time spent by the commuter is minimum.

(b) Time spent driving at $r_1 = \frac{y}{r_1}$

$$= \frac{\sqrt{\frac{r_1 c}{r_2}}}{r_1} = \sqrt{\frac{c}{r_1 r_2}}$$

Time spent driving at $r_2 = \dfrac{x}{r_2}$

$$= \frac{\sqrt{\frac{r_2 c}{r_1}}}{r_2} = \sqrt{\frac{c}{r_1 r_2}}$$

Therefore, equal time is spent driving at r_1 and r_2.

48. (a) $C(v) = avp + b\dfrac{p}{v}$

$$C'(v) = ap + \left(\frac{-bp}{v^2}\right)$$

$$C'(v) = 0 \Rightarrow ap + \left(\frac{-bp}{v^2}\right) = 0$$

$$a + \left(\frac{-b}{v^2}\right) = 0$$

$$a = \frac{b}{v^2}$$

$$v^2 = \frac{b}{a}$$

$$v = \pm\sqrt{\frac{b}{a}}$$

$$C''(v) = \frac{bp}{v^3}$$

$C''(v) > 0$ at $v = \sqrt{\dfrac{b}{a}}$.

Therefore, $v = \sqrt{\frac{b}{a}}$ to minimize $C(v)$.

(b) $C(v) = ap\dfrac{v^2}{v - v_c} + b\dfrac{p}{v - v_c}$

$$C'(v) = ap\left[\frac{(v - v_c)(2v) - v^2}{(v - v_c)^2}\right] + \left[\frac{-bp}{(v - v_c)^2}\right]$$

$$= \frac{1}{(v - v_c)^2}\left[2apv(v - v_c) - apv^2 - bp\right]$$

$$= \frac{1}{(v - v_c)^2}\left[apv^2 - 2apv_c v - bp\right]$$

$C'(v) = 0 \Rightarrow$

$apv^2 - 2apv_c v - bp = 0$

$$v = \frac{2apv_c \pm \sqrt{(2apv_c)^2 + 4abp^2}}{2ap}$$

$$v = v_c \pm \sqrt{{v_c}^2 + \frac{b}{a}}$$

Therefore, $v = v_c \pm \sqrt{{v_c}^2 + \dfrac{b}{a}}$ minimizes $C(v)$

Ch. 3 Review Exercises

1. $f(x) = e^{3x}$, $x_0 = 0$,
$f'(x) = 3e^{3x}$
$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(0) + f'(0)(x - 0) \\ &= e^{3\cdot 0} + 3e^{3\cdot 0}x \\ &= 1 + 3x \end{aligned}$$

2. $f'(x) = \dfrac{2x}{2\sqrt{x^2 + 3}}$.
$f(1) = 2$, and $f'(1) = 1/2$.
$L(x) = \frac{1}{2}(x - 1) + 2$.

3. $f(x) = \sqrt[3]{x} = x^{1/3}$, $x_0 = 8$
$f'(x) = \frac{1}{3}x^{-2/3}$
$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(8) + f'(8)(x - 8) \\ &= \sqrt[3]{8} + \tfrac{1}{3}(8)^{-2/3}(x - 8) \\ &= 2 + \tfrac{1}{12}(x - 8) \end{aligned}$$
$L(7.96) = 2 + \frac{1}{12}(7.96 - 8) \approx 1.99666$

4. $\sin 3$ is close to $\sin\pi$. If $y = \sin x$, $y' = \cos x$. The point is $(\pi, 0)$ and the slope is -1. The linear approximation of $\sin x$ at $x = \pi$ is
$L(x) = -(x - \pi)$, so
$\sin 3 \approx -(3 - \pi) \approx 0.14159$.

5. From the graph of $f(x) = x^3 + 5x - 1$, there is one root.
$f'(x) = 3x^2 + 5$
Starting with $x_0 = 0$, Newton's method gives $x_1 = 0.2$, $x_2 = 0.198437$, and $x_3 = 0.198437$.

6. From the graph of $f(x) = x^3 - e^{-x}$, there is one root.
$f'(x) = 3x^2 + e^{-x}$
Starting with $x_0 = 1$, Newton's method gives $x_1 = 0.8123$, $x_2 = 0.7743$, and $x_3 = 0.7729$, which is accurate to 4 decimal places.

7. Near an inflection point, the rate of change of the rate of change of $f(x)$ is very small so there aren't any big dropoffs or sharp increases nearby to make the linear approximation inaccurate.

8. If $y = \dfrac{1}{1 - x}$, then $y' = \dfrac{1}{(1 - x)^2}$.

For "small" x, x is near 0. The point on the curve when $x = 0$ is $(0, 1)$, and the slope is 1, so the linear approximation is $L(x) = x + 1$, and this is valid for "small" x.

9. $\displaystyle\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$ is type $\frac{0}{0}$;

L'Hôpital's Rule gives

$$\lim_{x\to 1}\frac{3x^2}{2x}=\frac{3}{2}.$$

10. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x^2+3x}$ is type $\frac{0}{0}$;
L'Hôpital's Rule gives

$$\lim_{x\to 0}\frac{\cos x}{2x+3}=\frac{1}{3}.$$

11. $\displaystyle\lim_{x\to 0}\frac{e^{2x}}{x^4+2}$ is type $\frac{\infty}{\infty}$;
applying L'Hôpital's Rule twice gives:

$$\begin{aligned}&\lim_{x\to\infty}\frac{2e^{2x}}{4x^3}\\ &=\lim_{x\to\infty}\frac{4e^{2x}}{12x^2}=\lim_{x\to\infty}\frac{8e^{2x}}{24x}\\ &=\lim_{x\to\infty}\frac{16e^{2x}}{24}=\infty\end{aligned}$$

12. $\displaystyle\lim_{x\to\infty}(x^2e^{-3x})=\lim_{x\to\infty}\frac{x^2}{e^{3x}}$ is type $\frac{\infty}{\infty}$;
applying L'Hôpital's Rule twice gives:

$$\begin{aligned}&\lim_{x\to\infty}\frac{2x}{3e^{3x}}\\ &=\lim_{x\to\infty}\frac{2}{9e^{3x}}=0\end{aligned}$$

13.

$$\begin{aligned}L&=\lim_{x\to 2^+}\left|\frac{x+1}{x-2}\right|^{\sqrt{x^2-4}}\\ \ln L&=\lim_{x\to 2^+}\left(\sqrt{x^2-4}\ln\left|\frac{x+1}{x-2}\right|\right)\\ &=\lim_{x\to 2^+}\left(\frac{\ln\left|\frac{x+1}{x-2}\right|}{(x^2-4)^{-1/2}}\right)\\ &=\lim_{x\to 2^+}\left(\frac{\left|\frac{x-2}{x+1}\right|\frac{-3}{(x-2)^2}}{-x(x^2-4)^{-3/2}}\right)\\ &=\lim_{x\to 2^+}\left(\frac{3(x^2-4)^{3/2}}{x(x+1)(x-2)}\right)\\ &=\lim_{x\to 2^+}\left(\frac{3(x-2)^{1/2}(x+2)^{3/2}}{x(x+1)}\right)\\ \ln L&=0\\ L&=1\end{aligned}$$

14. $\displaystyle\lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$
is type $\frac{0}{0}$ so we can apply L'Hôpital's Rule:

$$\begin{aligned}&\lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{1}{x}\right)}(-x^{-2})}{-x^{-2}}\\ &=\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)}=1\end{aligned}$$

15.

$$\begin{aligned}\lim_{x\to 0^+}(\tan x\ln x)&=\lim_{x\to 0^+}\left(\frac{\ln x}{\cot x}\right)\\ &=\lim_{x\to 0^+}\left(\frac{1/x}{-\csc^2 x}\right)\\ &=\lim_{x\to 0^+}-\left(\frac{\sin^2 x}{x}\right)\\ &=-\lim_{x\to 0^+}\left(\frac{\sin x}{x}\sin x\right)\\ &=(-1)(0)=0\end{aligned}$$

16. $\displaystyle\lim_{x\to 0}\frac{\tan^{-1}x}{\sin^{-1}x}$ is type $\frac{0}{0}$;
we can apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}}=\lim_{x\to 0}\frac{\sqrt{1-x^2}}{1+x^2}=1$$

17. $f'(x)=3x^2+6x-9=3(x^2+2x-3)$
$=3(x+3)(x-1)$
So the critical numbers are $x=1$ and $x=-3$.
$f'(x)>0$ on $(-\infty,-3)\cup(1,\infty)$
$f'(x)<0$ on $(-3,1)$
Hence f is increasing on $(-\infty,-3)$ and on $(1,\infty)$ and f is decreasing on $(-3,1)$. Thus there is a local max at $x=-3$ and a local min at $x=1$.

$f''(x)=3(2x+2)=6(x+1)$
$f''(x)>0$ on $(-1,\infty)$
$f''(x)<0$ on $(-\infty,-1)$
Hence f is concave up on $(-1,\infty)$ and concave down on $(-\infty,-1)$, and there is an inflection point at $x=-1$.

18. $f'(x)=4x^3-4$
$f'(x)=0$ when $x=1$, and this is the only critical number. The function is decreasing for $x<1$ and increasing for $x>1$.
$f''=12x^2>0$ when $x=1$, so this is a local minimum. $f''=0$ when $x=0$, but does not change sign there, so there are no inflection points. The function is concave up everywhere.

19. $f'(x)=4x^3-12x^2=4x^2(x-3)$
$x=0,3$ are critical numbers.
$f'(x)>0$ on $(3,\infty)$
$f'(x)<0$ on $(-\infty,0)\cup(0,3)$
f increasing on $(3,\infty)$, decreasing on $(-\infty,3)$
so $x=3$ is a local min.
$f''(x)=12x^2-24x=12x(x-2)$

$f''(x) > 0$ on $(-\infty, 0) \cup (2, \infty)$
$f''(x) < 0$ on $(0, 2)$
f is concave up on $(-\infty, 0) \cup (2, \infty)$, concave down on $(0, 2)$ so $x = 0$, 2 are inflection points.

20. $f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2)$
$f'(x) = 0$ when $x = 4$ and $x = -2$. The function is increasing for $x < -2$, then decreasing for $-2 < x < 4$, and increasing for $x > 4$. $x = -2$ represents a local maximum, and $x = 4$ represents a local minimum.
$f''(x) = 6x - 6$
$f''(x) = 0$ when $x = 1$, and changes sign there, so $x = 1$ is an inflection point. The function is concave down for $x < 1$ and concave up for $x > 1$.

21. $f'(x) = e^{-4x} + xe^{-4x}(-4) = e^{-4x}(1 - 4x)$
$x = 1/4$ is a critical number.
$f'(x) > 0$ on $\left(-\infty, \frac{1}{4}\right)$
$f'(x) < 0$ on $\left(\frac{1}{4}, \infty\right)$
f increasing on $\left(-\infty, \frac{1}{4}\right)$, decreasing on $\left(-\frac{1}{4}, \infty\right)$ so $x = 1/4$ is a local max.
$f''(x) = e^{-4x}(-4)(1-4x) + e^{-4x}(-4)$
$\quad = -4e^{-4x}(2 - 4x)$
$f''(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$
$f''(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$
f is concave up on $\left(\frac{1}{2}, \infty\right)$, concave down on $\left(-\infty, \frac{1}{2}\right)$ so $x = 1/2$ is inflection point.

22. $f'(x) = 2x \ln x + x = x(2\ln x + 1)$
$f'(x) = 0$ when $\ln x = -1/2$, so $x = e^{-1/2}$. ($x = 0$ is not a critical number because it is not in the domain of the function.) The function is decreasing for $0 < x < e^{-1/2}$, and increasing for $x > e^{-1/2}$. The critical number $x = e^{-1/2}$ represents a minimum.
$f''(x) = 2\ln x + 3$
$f''(x) = 0$ when $x = e^{-3/2}$ and the sign changes from negative to positive there, so this is an inflection point. The function is concave down for $0 < x < e^{-3/2}$ and concave up for $x > e^{-3/2}$.

23. $f'(x) = \dfrac{x^2 - (x-90)(2x)}{x^4}$
$\quad = \dfrac{-(x-180)}{x^3}$
$x = 180$ is the only critical number.
$f'(x) < 0$ on $(-\infty, 0) \cup (180, \infty)$
$f'(x) > 0$ on $(0, 180)$
$f(x)$ is decreasing on $(-\infty, 0) \cup (180, \infty)$ and increasing on $(0, 180)$ so $f(x)$ has a local maximum at $x = 180$.
$f''(x) = -\dfrac{x^3 - (x-180)(3x^2)}{x^6}$
$\quad = -\dfrac{-2x + 540}{x^4}$
$f''(x) < 0$ on $(-\infty, 0) \cup (0, 270)$
$f''(x) > 0$ on $(270, \infty)$ so $x = 90$ is an inflection point.

24. $f'(x) = \dfrac{4x}{3(x^2-1)^{1/3}}$
$f'(x) = 0$ at $x = 0$ and is undefined at $x = \pm 1$. The function is decreasing for $x < -1$, increasing for $-1 < x < 0$, decreasing for $0 < x < 1$, and increasing for $1 < x$. Critical numbers $x = \pm 1$ are minima, and $x = 0$ is a maximum.
$f''(x) = \dfrac{4(x^2-3)}{9(x^2-1)^{4/3}}$
$f''(x) = 0$ when $x = \pm\sqrt{3}$, and undefined for $x = \pm 1$. The function is concave up for $x < -\sqrt{3}$, concave down for $-\sqrt{3} < x < -1$, concave down for $-1 < x < 1$, concave down for $1 < x < \sqrt{3}$, and concave up for $\sqrt{3} < x$. The inflection points are $x = \pm\sqrt{3}$.

25. $f'(x) = \dfrac{x^2 + 4 - x(2x)}{(x^2+4)^2}$
$\quad = \dfrac{4 - x^2}{(x^2+4)^2}$
$x = \pm 2$ are critical numbers.
$f'(x) > 0$ on $(-2, 2)$
$f'(x) < 0$ on $(-\infty, -2) \cup (2, \infty)$
f increasing on $(-2, 2)$, decreasing on $(-\infty, -2)$ and on $(2, \infty)$ so f had a local min at $x = -2$ and a local max at $x = 2$.
$f''(x) =$
$\dfrac{-2x(x^2+4)^2 - (4-x^2)[2(x^2+4)\cdot 2x]}{(x^2+4)^4}$
$\quad = \dfrac{2x^3 - 24x}{(x^2+4)^3}$
$f''(x) > 0$ on $\left(-\sqrt{12}, 0\right) \cup \left(\sqrt{12}, \infty\right)$
$f''(x) < 0$ on $\left(-\infty, -\sqrt{12}\right) \cup \left(0, \sqrt{12}\right)$
f is concave up on $\left(-\sqrt{12}, 0\right) \cup \left(\sqrt{12}, \infty\right)$, concave down on $\left(-\infty, -\sqrt{12}\right) \cup \left(0, \sqrt{12}\right)$ so $x = \pm\sqrt{12}$, 0 are inflection points.

26. $f'(x) = \dfrac{2}{(x^2+4)^{3/2}}$
$f'(x)$ is never zero and is defined for all x, so there are no critical numbers. The function is increasing for all x.
$f''(x) = \dfrac{-6x}{(x^2+4)^{5/2}}$
$f''(x) = 0$ when $x = 0$. The function is concave up for $x < 0$, concave down for $x > 0$, and the inflection point is $x = 0$.

27. $f'(x) = 3x^2 + 6x - 9$
$\quad = 3(x+3)(x-1)$

$x = -3$, $x = 1$ are critical numbers, but $x = -3 \notin [0, 4]$.
$f(0) = 0^3 + 3 \cdot 0^2 - 9 \cdot 0 = 0$
$f(4) = 4^3 + 3 \cdot 4^2 - 9 \cdot 4 = 76$
$f(1) = 1^3 + 3 \cdot 1^2 - 9 \cdot 1 = -5$
So $f(4) = 76$ is absolute max on $[0, 4]$, $f(1) = -5$ is absolute min.

28. First note that $f(x) = \sqrt{x(x-1)(x-2)}$ is only defined on $[0, 1] \cup [2, \infty)$. So we are looking at the intervals $[0, 1] \cup [2, 3]$.

$$f'(x) = \frac{3x^2 - 6x + 2}{2\sqrt{x^3 - 3x^2 + 2x}}$$

The numerator has roots $x = \frac{3 \pm \sqrt{3}}{3}$, but $f(x)$ is only defined at $\frac{3-\sqrt{3}}{3}$. The denominator has zeros at $x = 0$, 1 and 2. Plus we have to check the values of f at the endpoint $x = 3$. We find:
$f(0) = 0$
$f(\frac{3-\sqrt{3}}{3}) \approx 0.6204$
$f(1) = 0$
$f(2) = 0$
$f(3) = \sqrt{6} \approx 2.4495$
Thus $f(x)$ has an absolute maximum on this interval at $x = 3$ and absolute minimums at $x = 0$, $x = 1$ and $x = 2$.

29. $f'(x) = \frac{4}{5}x^{-1/5}$
$x = 0$ is critical number.
$f(-2) = (-2)^{4/5} \approx 1.74$
$f(3) = (3)^{4/5} \approx 2.41$
$f(0) = (0)^{4/5} = 0$
$f(0) = 0$ is absolute min, $f(3) = 3^{4/5}$ is absolute max.

30. $f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$
$f'(x) = 0$ when $x = 0$ and $x = 2$. We test $f(x)$ at the critical numbers in the interval $[-1, 4]$, and the endpoints.
$f(-1) = e \approx 2.718$
$f(0) = 0$
$f(2) = 4/e^2 \approx 0.541$
$f(4) = 16/e^4 \approx 0.293$
The absolute maximum is $f(-1) = e$, and the absolute minimum is $f(0) = 0$.

31. $f'(x) = 3x^2 + 8x + 2$

$f'(x) = 0$ when

$$x = \frac{-8 \pm \sqrt{64 - 24}}{6} = -\frac{4}{3} \pm \frac{\sqrt{10}}{3}$$

$x = -\dfrac{4}{3} - \dfrac{\sqrt{10}}{3}$ is local max, $x = -\dfrac{4}{3} + \dfrac{\sqrt{10}}{3}$ is local min.

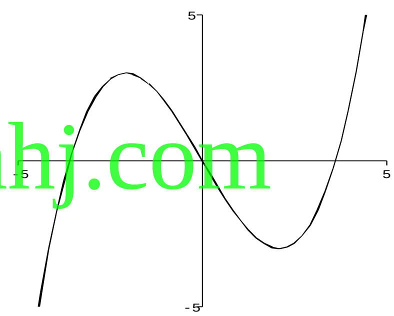
32. $f'(x) = 4x^3 - 6x + 2$
$\quad = 2(x - 1)(2x^2 + 2x - 1)$

$f'(x) = 0$ when $x = 1$ and $x = \dfrac{-1 \pm \sqrt{3}}{2}$, and the derivative changes sign at these values, so these critical numbers are all extrema.

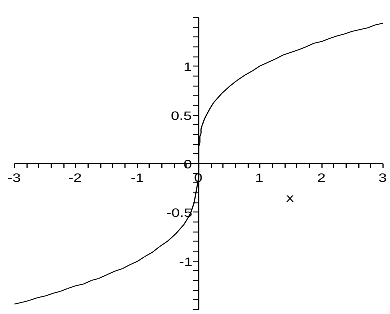
33. $f'(x) = 5x^4 - 4x + 1 = 0$
$x \approx 0.2553,\ 0.8227$
local min at $x \approx 0.8227$,
local max at $x \approx 0.2553$.

34. $f'(x) = 5x^4 + 8x - 4$
$f'(x) = 0$ at approximately $x = -1.3033$ and $x = 0.4696$ (found using Newton's method, or a CAS numerical solver). The derivative changes sign at these values so they correspond to extrema: $x = -1.3033$ is a local max and $x = 0.4696$ is a local min.

35. One possible graph:

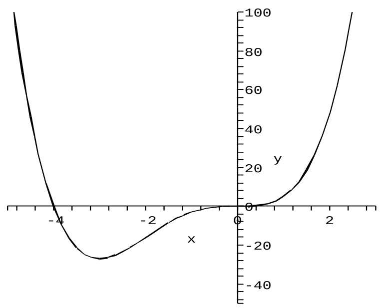


36. One possible graph:

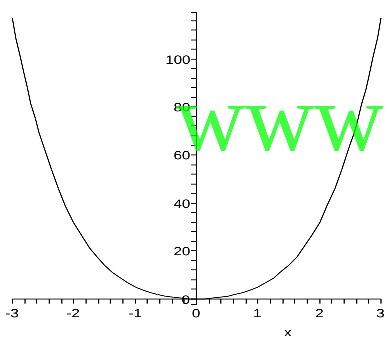


37. $f'(x) = 4x^3 + 12x^2 = 4x^2(4x + 3)$
$f''(x) = 12x^2 + 24x = 12x(x + 2)$
$f'(x) > 0$ on $(-3, 0) \cup (0, \infty)$
$f'(x) < 0$ on $(-\infty, -3)$
$f''(x) > 0$ on $(-\infty, -2) \cup (0, \infty)$
$f''(x) < 0$ on $(-2, 0)$
f increasing on $(-3, \infty)$, decreasing on $(-\infty, -3)$, concave up on $(-\infty, -2) \cup (0, \infty)$, concave down on $(-2, 0)$, local min at $x = -3$,

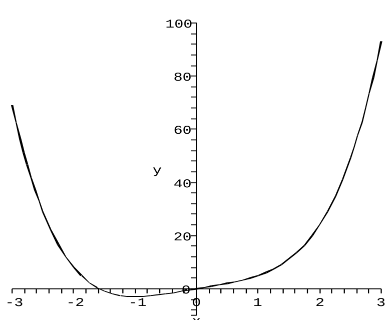
inflection points at $x = -2,\ 0$.
$f(x) \to \infty$ as $x \to \pm\infty$.



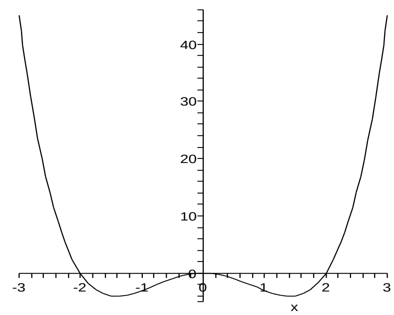
38. $f'(x) = 4x^3 + 8x$
$f'(x) = 0$ when $x = 0$.
$f'' = 12x^2 + 8 > 0$ at $x = 0$, so this is a minimum. $f''(x) > 0$ for all x so there are no inflection points.
$f(x) \to \infty$ as $x \to \pm\infty$.



39. $f'(x) = 4x^3 + 4 = 4(x^3 + 1)$
$f''(x) = 12x^2$
$f'(x) > 0$ on $(-1, \infty)$
$f'(x) < 0$ on $(-\infty, -1)$
$f''(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$
f increasing on $(-1, \infty)$, decreasing on $(-\infty, -1)$, concave up on $(-\infty, \infty)$, local min at $x = -1$.
$f(x) \to \infty$ as $x \to \pm\infty$.



40. $f'(x) = 4x^3 - 8x$
$f'(x) = 0$ when $x = 0$ and $x = \pm\sqrt{2}$.
$f'' = 12x^2 - 8 < 0$ at $x = 0$, so this is a maximum. $f''(x) > 0$ for $x = \pm\sqrt{2}$, so these are minima.
$f''(x) = 0$ when $x = \pm\sqrt{2/3}$, and changes sign there, so these are inflection points.
$f(x) \to \infty$ as $x \to \pm\infty$.



41. $f'(x) = \dfrac{x^2 + 1 - x(2x)}{(x^2+1)^2}$
$= \dfrac{1 - x^2}{(x^2+1)^2}$

$f''(x) =$
$\dfrac{-2x(x^2+1)^2 - (1-x^2)2(x^2+1)2x}{(x^2+1)^4}$
$= \dfrac{2x(x^2-3)}{(x^2+1)^4}$
$f'(x) > 0$ on $(-1, 1)$
$f'(x) < 0$ on $(-\infty, -1) \cup (1, \infty)$
$f''(x) > 0$ on $\left(-\sqrt{3}, 0\right) \cup \left(\sqrt{3}, \infty\right)$
$f''(x) < 0$ on $\left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right)$
f increasing on $(-1, 1)$, decreasing on $(-\infty, -1)$ and on $(1, \infty)$, concave up on

$$\left(-\sqrt{3}, 0\right) \cup \left(\sqrt{3}, \infty\right),$$

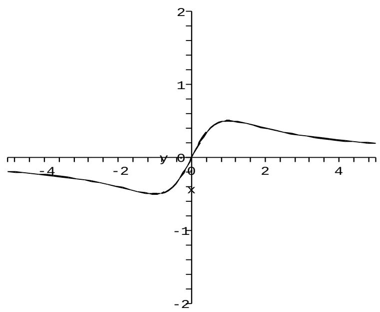
concave down on

$$\left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right),$$

local min at $x = -1$, local max at $x = 1$, inflection points at $0, \pm\sqrt{3}$.

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{x}{x^2+1} = 0$$

So f has a horizontal asymptote at $y = 0$.

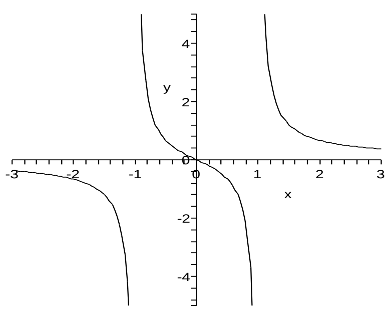


42. $f'(x) = -\dfrac{x^2+1}{(x^2-1)^2}$

is undefined when $f(x)$ is undefined, and is never zero. There are no extrema. There are vertical asymptotes at $x = \pm 1$, and horizontal asymptote $y = 0$.

$$f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

$f''(x) = 0$ when $x = 0$, and this is the inflection point: $f(x)$ is concave down on $(-\infty, -1)$ and $(0, 1)$; $f(x)$ is concave up on $(-1, 0)$ and $(1, \infty)$.



43. $f'(x) = \dfrac{(2x)(x^2+1) - x^2(2x)}{(x^2+1)^2}$

$= \dfrac{2x}{(x^2-1)^2}$

$f''(x) = \dfrac{2(x^2+1)^2 - 2x \cdot 2(x^2+1)2x}{(x^2+1)^4}$

$= \dfrac{2-6x^2}{(x^2+1)^3}$

$f'(x) > 0$ on $(0, \infty)$

$f'(x) < 0$ on $(-\infty, 0)$

$f''(x) > 0$ on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$

$f''(x) < 0$ on $\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right)$

f increasing on $(0, \infty)$ decreasing on $(-\infty, 0)$, concave up on

$$\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right),$$

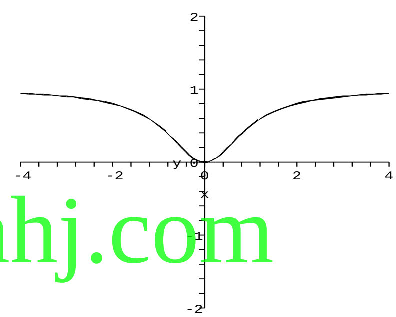
concave down on

$$\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right),$$

local min at $x = 0$, inflection points at $x = \pm\sqrt{1/3}$.

$$\lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} = 1$$

So f has a horizontal asymptote at $y = 1$.

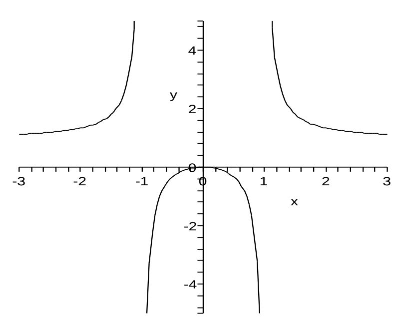


44. $f'(x) = -\dfrac{2x}{(x^2-1)^2}$

$f'(x) = 0$ when $x = 0$, and is undefined when $f(x)$ is undefined. There is a local maximum at $x = 0$. There are vertical asymptotes at $x = \pm 1$, and horizontal asymptote $y = 1$.

$$f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$$

$f''(x) \neq 0$ for any x, and there are no inflection points: $f(x)$ is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$.



45. $f'(x) = \dfrac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2}$

$= \dfrac{x^4 - 3x^2}{(x^2-1)^2}$

$f''(x) = \dfrac{(4x^3 - 6x)(x^2-1)^2}{(x^2-1)^4}$

$- \dfrac{(x^4 - 3x^2)2(x^2-1)2x}{(x^2-1)^4}$

$= \dfrac{2x^3 + 6x}{(x^2-1)^4}$

$f'(x) > 0$ on $\left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, \infty\right)$

$f'(x) < 0$ on $\left(-\sqrt{3}, -1\right) \cup (-1, 0) \cup (0, 1) \cup \left(1, \sqrt{3}\right)$

$f''(x) > 0$ on $(-1, 0) \cup (1, \infty)$

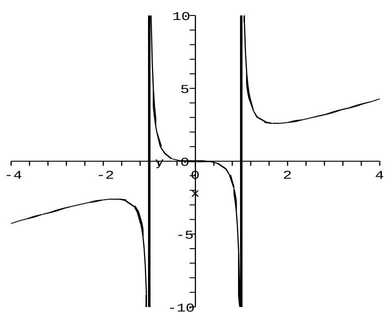
$f''(x) < 0$ on $(-\infty, -1) \cup (0, 1)$

f increasing on $(-\infty, -\sqrt{3})$ and on $(\sqrt{3}, \infty)$; decreasing on $(-\sqrt{3}, -1)$ and on $(-1, 1)$ and on $(1, \sqrt{3})$; concave up on $(-1, 0) \cup (1, \infty)$, concave down on $(-\infty, -1) \cup (0, 1)$; $x = -\sqrt{3}$ local max; $x = \sqrt{3}$ local min; $x = 0$ inflection point. f is undefined at $x = -1$ and $x = 1$.

$\lim\limits_{x \to 1^+} \dfrac{x^3}{x^2 - 1} = \infty$, and

$\lim\limits_{x \to 1^-} \dfrac{x^3}{x^2 - 1} = -\infty$

So f has vertical asymptotes at $x = 1$ and $x = -1$.

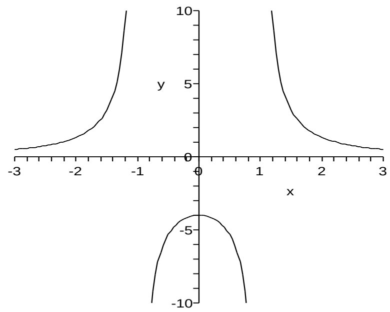


46. $f'(x) = -\dfrac{8x}{(x^2-1)^2}$

$f'(x) = 0$ when $x = 0$, and is undefined when $f(x)$ is undefined. $f(x)$ is increasing on $(-\infty, -1)$ and $(-1, 0)$; $f(x)$ is decreasing on $(0, 1)$ and $(1, \infty)$. There is a local maximum at $x = 0$. There are vertical asymptotes at $x = \pm 1$, and horizontal asymptote $y = 0$.

$$f''(x) = \frac{8(3x^2 + 1)}{(x^2-1)^3}$$

$f''(x) \neq 0$ for any x, and there are no inflection points. $f(x)$ is concave up on $(-\infty, -1)$ and $(1, \infty)$; $f(x)$ is concave down on $(-1, 1)$.



47. $d = \sqrt{(x-2)^2 + (y-1)^2}$

$= \sqrt{(x-2)^2 + (2x^2-1)^2}$

$f(x) = (x-2)^2 + (2x^2-1)^2$

$f'(x) = 2(x-2) + 2(2x^2-1)4x$

$= 16x^3 - 6x - 4$

$f'(x) = 0$ when $x \approx 0.8237$

$f'(x) < 0$ on $(-\infty, 0.8237)$

$f'(x) > 0$ on $(0.8237, \infty)$

So $x \approx 0.8237$ corresponds to the closest point.

$y = 2x^2 = 2(0.8237)^2 = 1.3570$

$(0.8237, 1.3570)$ is closest to $(2, 1)$.

48. We compute the slope of the tangent line to $y = 2x^2$ at the closest point $(0.8237, 1.3570)$. When $x = 0.8237$, we get $y' = 3.2948$. The slope of the line between $(2, 1)$ and $(0.8237, 1.3570)$ is

$$\frac{1 - 1.3570}{2 - 0.8237} = -0.3035 = \frac{-1}{3.2948},$$

so the lines are perpendicular.

49. $C(x) = 6\sqrt{4^2 + (4-x)^2} + 2\sqrt{2^2 + x^2}$

$C'(x) =$

$6 \cdot \frac{1}{2}[16 + (4-x)^2]^{-1/2} \cdot 2(4-x)(-1)$

$+ 2\frac{1}{2}(4 + x^2)^{-1/2} \cdot 2x$

$= \dfrac{6(x-4)}{\sqrt{16 + (4-x)^2}} + \dfrac{2x}{\sqrt{4 + x^2}}$

$C'(x) = 0$ when $x \approx 2.864$

$C'(x) < 0$ on $(0, 2.864)$

$C'(x) > 0$ on $(2.864, 4)$

So $x \approx 2.864$ gives the minimum cost. Locate highway corner $4 - 2.864 = 1.136$ miles east of point A.

50. Let $F(v) = e^{-v/2}$. Then $F'(v) = -0.5e^{-v/2}$, so $F'(v) < 0$ for all v. Thus $F(v)$ is decreasing for all v. This says that as the speed of contraction increases, the force produced decreases.

Let $P(v) = ve^{-v/2}$. Then
$P'(v) = e^{-v/2}(1 - \frac{1}{2}v)$.

$P'(v) = 0$ when $v = 2$. We check that $P'(0) > 0$ and $P'(4) < 0$ so $v = 2$ is in fact a maximum.

51. Area: $A = 2\pi r^2 + 2\pi rh$
Convert to in^3:
$$\begin{aligned}16 \text{ fl oz} &= 16 \text{ fl oz} \cdot 1.80469 \text{ in}^3/\text{fl oz}\\ &= 28.87504\text{in}^3\end{aligned}$$
Volume: $V = \pi r^2 h$
$$h = \frac{\text{Vol}}{\pi r^2} = \frac{28.87504}{\pi r^2}$$
$$A(r) = 2\pi\left(r^2 + \frac{28.87504}{\pi r}\right)$$
$$A'(r) = 2\pi\left(2r - \frac{28.87504}{\pi r^2}\right)$$
$$2\pi r^3 = 28.87504$$
$$r = \sqrt[3]{\frac{28.87504}{2\pi}} \approx 1.663$$
$A'(r) < 0$ on $(0, 1.663)$
$A'(r) > 0$ on $(1.663, \infty)$

So $r \approx 1.663$ gives the minimum surface area.
$$h = \frac{28.87504}{\pi(1.663)^2} \approx 3.325$$

52. If $C(x) = 0.02x^2 + 4x + 1200$, then $C'(x) = 0.04x + 4 > 0$ for positive values of x (number of items manufactured). This must be positive because the cost function must be increasing. It must cost more to manufacture more items.
$C''(x) = 0.04 > 0$. This means that the cost per item is rising as the number of items produced increases. (For an efficient process, the cost per item should decrease as the number of items increases.)

53. Let θ_1 be the angle from the horizontal to the upper line segment defining θ and let θ_2 be the angle from the horizontal to the lower line segment defining θ. Then the length of the side opposite θ_2 is $\frac{H-P}{2}$ while the length of the side opposite θ_1 is $\frac{H+P}{2}$. Then
$$\begin{aligned}\theta(x) &= \theta_1 - \theta_2\\ &= \tan^{-1}\left(\frac{H+P}{2x}\right)\\ &\quad - \tan^{-1}\left(\frac{H-P}{2x}\right)\end{aligned}$$
and so
$$\begin{aligned}\theta'(x) = &\frac{1}{1+\left(\frac{H+P}{2x}\right)^2}\left(-\frac{H+P}{2x^2}\right)\\ &- \frac{1}{1+\left(\frac{H-P}{2x}\right)^2}\left(-\frac{H-P}{2x^2}\right).\end{aligned}$$
We set this equal to 0:
$$0 = \frac{-2(H+P)}{4x^2+(H+P)^2} + \frac{2(H-P)}{4x^2+(H-P)^2}$$
and solve for x:
$$\begin{aligned}&\frac{2(H+P)}{4x^2+(H+P)^2} = \frac{2(H-P)}{4x^2+(H-P)^2}\\ &8x^2(H+P) - 8x^2(H-P)\\ &\quad = 2(H-P)(H+P)^2\\ &\quad\quad - 2(H+P)(H-P)^2\\ &8x^2(2P) = 2(H-P)(H+P)(2P)\\ &x^2 = \frac{H^2-P^2}{4}\\ &x = \frac{\sqrt{H^2-P^2}}{2}.\end{aligned}$$

54. From exercise 53 we know that
$$\theta'(x) = \frac{-2(H+P)}{4x^2+(H+P)^2} + \frac{2(H-P)}{4x^2+(H-P)^2}$$
and that the function $\theta(x)$ is maximized at
$$x = \frac{\sqrt{H^2-P^2}}{2}.$$

Plugging in the appropriate H and P values for high school shows that $\theta(x)$ is maximized by $x \approx 23.9792$. This is not in the range specified. In order to find out whether $\theta(x)$ is increasing or decreasing in the interval specified we plug the H and P values into the expression for $\theta'(x)$ and then plug in a value in our interval, say 55. We find that $\theta'(55) \approx -0.00392$. Since this is negative, $\theta(x)$ is decreasing on this interval, so the announcers must be wrong.

Following the same procedure for college, we find that $\theta(x)$ is maximized by $x \approx 17.7324$ and $\theta'(55) \approx -0.00412$ so again the announcers would be wrong.

Finally, for pros we see that $\theta(x)$ is maximized at $x = 0$ and $\theta'(55) \approx -0.0055$ so the announcers would be wrong once again. In this situation there is no x value for which the announcers would be correct, but in the high school and college situations, if the field goal is taken from some x less than the x which maximized $\theta(x)$, the announcers would be correct.

55. $Q'(t) = -3e^{-3t}\sin 2t + e^{-3t}\cos 2t \cdot 2$
$= e^{-3t}(2\cos 2t - 3\sin 2t)$ amps

56. $f(x) = 0.3x(4-x)$, $f'(x) = 1.2-0.6x = 0$ when $x = 2$, and changes from positive to negative there, so this represents a maximum.

57. $\rho(x) = m'(x) = 2x$
As you move along the rod to the right, its density increases.

58. With no studying, the person scores $f(0) = \dfrac{90}{1+4} = 18$.
$f'(x) = \dfrac{144e^{-0.4t}}{(1+4e^{-0.4t})^2}$.

If the student were to study one hour, the score will increase by approximately

$f'(0) = \dfrac{144}{25} = 5.76$ points.

59. $C'(x) = 0.04x + 20$
$C'(20) = 0.04(20) + 20 = 20.8$
$C(20) - C(19) =$
$0.02(20)^2 + 20(20) + 1800$
$- [0.02(19)^2 + 20(19) + 1800]$
$= 20.78$

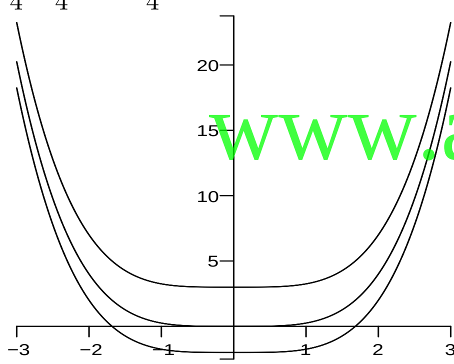
60. $\overline{C}(x) = \dfrac{0.02x^2 + 20x + 1800}{x}$
$= 0.02x + 20 + \dfrac{1800}{x}$,
$\overline{C}'(x) = 0.02 - \dfrac{1800}{x^2}$
$\overline{C}'(x) = 0$ when $x = 300$, and the derivative changes from negative to positive here, so $x = 300$ gives the minimum average cost.

Chapter 4

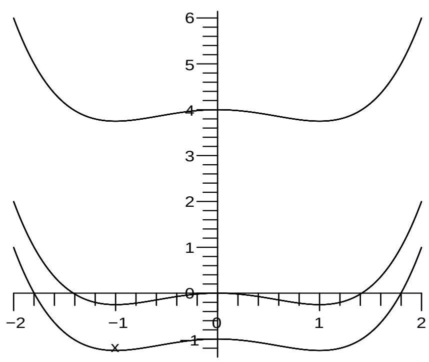
Integration

4.1 Antiderivatives

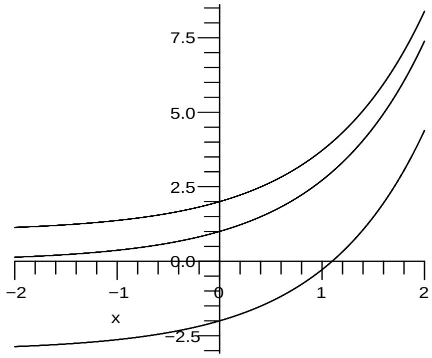
1. $\dfrac{x^4}{4}, \dfrac{x^4}{4}+3, \dfrac{x^4}{4}-2$



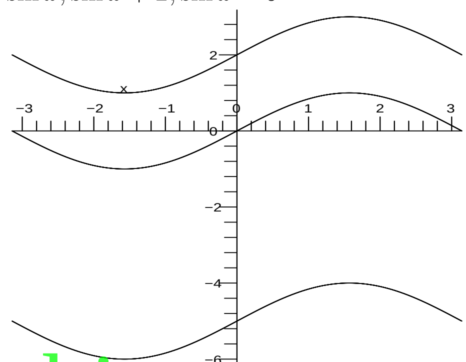
2. $\dfrac{x^4}{4}-\dfrac{x^2}{2}, \dfrac{x^4}{4}-\dfrac{x^2}{2}-1, \dfrac{x^4}{4}-\dfrac{x^2}{2}+4$



3. e^x, e^x+1, e^x-3



4. $\sin x, \sin x+2, \sin x-5$



5. $\displaystyle\int (3x^4-3x)dx = \frac{3}{5}x^5 - \frac{3}{2}x^2 + c$

6. $\displaystyle\int (x^3-2)dx = \frac{1}{4}x^4 - 2x + c$

7. $\displaystyle\int \left(3\sqrt{x} - \frac{1}{x^4}\right) dx = 2x^{3/2} + \frac{x^{-3}}{3} + c$

8. $\displaystyle\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right) dx$
$= -2x^{-1} + 2x^{1/2} + c$

9. $\displaystyle\int \frac{x^{1/3}-3}{x^{2/3}}dx = \int (x^{-1/3} - 3x^{-2/3})dx$
$= \dfrac{3}{2}x^{2/3} - 9x^{1/3} + c$

10. $\displaystyle\int \frac{x+2x^{3/4}}{x^{5/4}}dx = \int (x^{-1/4} + 2x^{-1/2})dx$
$= \dfrac{4}{3}x^{3/4} + 4x^{1/2} + c$

11. $\displaystyle\int (2\sin x + \cos x)dx = -2\cos x + \sin x + c$

12. $\displaystyle\int (3\cos x - \sin x)dx = 3\sin x + \cos x + c$

13. $\displaystyle\int 2\sec x \tan x dx = 2\sec x + c$

14. $\displaystyle\int \frac{4}{\sqrt{1-x^2}}dx = 4\arcsin x + c$

15. $\displaystyle\int 5\sec^2 x dx = 5\tan x + c$

16. $\displaystyle\int \frac{4\cos x}{\sin^2 x}dx = -4\csc x + c$

17. $\displaystyle\int (3e^x - 2)dx = 3e^x - 2x + c$

18. $\displaystyle\int (4x - 2e^x)dx = 2x^2 - 2e^x + c$

19. $\displaystyle\int (3\cos x - 1/x)dx = 3\sin x - \ln|x| + c$

20. $\displaystyle\int (2x^{-1} + \sin x)dx = 2\ln|x| - \cos x + c$

21. $\displaystyle\int \frac{4x}{x^2+4}dx = 2\ln|x^2+4| + c$

22. $\displaystyle\int \frac{3}{4x^2+4}dx = \frac{3}{4}\tan^{-1} x + c$

23. $\displaystyle\int \frac{\cos x}{\sin x}dx = \ln|\sin x| + c$

24. $\displaystyle\int (2\cos x - e^x)dx = 2\sin x - e^x + c$

25. $\displaystyle\int \frac{e^x}{e^x+3}dx = \ln|e^x+3| + c$

26. $\displaystyle\int \frac{e^x+3}{e^x}dx = \int (1+3e^{-x})dx$
$= x - 3e^{-x} + c$

27. $\displaystyle\int x^{1/4}(x^{5/4}-4)dx = \int (x^{3/2} - 4x^{1/4})dx$
$\displaystyle= \frac{2}{5}x^{5/2} - \frac{16}{5}x^{5/4} + c$

28. $\displaystyle\int x^{2/3}(x^{-4/3}-3)dx = \int (x^{-2/3} - 3x^{2/3})dx$
$\displaystyle= 3x^{1/3} - \frac{9}{5}x^{5/3} + c$

29. $\displaystyle\frac{d}{dx}\ln|\sec x + \tan x|$
$\displaystyle= \frac{1}{\sec x + \tan x}\frac{d}{dx}(\sec x + \tan x)$
$\displaystyle= \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$
$\displaystyle= \frac{\sec x\,(\tan x + \sec x)}{\sec x + \tan x}$
$= \sec x$

30. $\displaystyle\frac{d}{dx}\ln|\sin x\cdot 2|$
$\displaystyle= \frac{1}{\sin x\cdot 2}\frac{d}{dx}(\sin x\cdot 2)$
$\displaystyle= \frac{2\cos x}{2\sin x} = \cot x$

31. (a) N/A
(b) By Power Formula,
$$\int (\sqrt{x^3}+4)dx = \frac{2}{5}x^{5/2} + 4x + c.$$

32. (a) By Power Formula,
$\displaystyle\int \frac{3x^2-4}{x^2}dx = \int (3-4x^{-2})dx$
$= 3x + 4x^{-1} + c$
(b) N/A

33. (a) N/A
(b) By Reversing derivative formula,
$\displaystyle\int \sec^2 x dx = \tan x + c$

34. (a) By Power Formula,
$\displaystyle\int \left(\frac{1}{x^2} - 1\right)dx = -\frac{1}{x} - x + c$
(b) N/A

35. Finding the antiderivative,
$\displaystyle f(x) = 3e^x + \frac{x^2}{2} + c.$

Since $f(0) = 4$,
we have $4 = f(0) = 3 + c$.
Therefore,
$\displaystyle f(x) = 3e^x + \frac{x^2}{2} + 1.$

36. Finding the antiderivative,
$f(x) = 4\sin x + c$.
Since $f(0) = 3$,
we have $3 = f(0) = c$.
Therefore,
$f(x) = 4\sin x + 3$.

37. Finding the antiderivative
$f'(x) = 4x^3 + 2e^x + c_1$.
Since, $f'(0) = 2$.
We have $2 = f'(0) = 2 + c_1$
and therefore
$f'(x) = 4x^3 + 2e^x$.
Finding the antiderivative,
$f(x) = x^4 + 2e^x + c_2$.
Since $f(0) = 3$,
We have $3 = f(0) = 2 + c_2$
Therefore,

$f(x) = x^4 + 2e^x + 1.$

38. Finding the antiderivative,
$f'(x) = 5x^4 + e^{2x} + c_1.$
Since $f'(0) = -3$,
we have $-3 = f'(0) = 1 + c_1$
Therefore,
$f'(x) = 5x^4 + e^{2x} - 4.$
Finding the antiderivative,
$f(x) = x^5 + \frac{e^{2x}}{2} - 4x + c_2.$
Since $f(0) = 2$,
We have $2 = f(0) = \frac{1}{2} + c_2$
Therefore,
$f(x) = x^5 + \frac{e^{2x}}{2} - 4x + \frac{3}{2}.$

39. Taking antiderivatives,
$f'(t) = 2t + t^2 + c_1$
$f(t) = t^2 + \frac{t^3}{3} + c_1 t + c_2$
Since $f(0) = 2$,
we have $2 = f(0) = c_2$
Therefore,
$f(t) = t^2 + \frac{t^3}{3} + c_1 t + 2.$
Since $f(3) = 2$,
we have
$2 = f(3) = 9 + 9 + 3c_1 + 2$
$-6 = c_1$
Therefore,
$f(t) = \frac{t^3}{3} + t^2 - 6t + 2.$

40. Taking antiderivatives,
$f'(t) = 4t + 3t^2 + c_1$
$f(t) = 2t^2 + t^3 + c_1 t + c_2$
Since $f(1) = 3$,
we have $3 = f(1) = 2 + 1 + c_1 + c_2$
Therefore,
$c_1 + c_2 = 0$
Since $f(-1) = -2$,
we have $-2 = f(-1) = 2 - 1 - c_1 + c_2$
Therefore, $-c_1 + c_2 = -3.$
So, $c_1 = \frac{3}{2}$ and $c_2 = -\frac{3}{2}$
Hence,
$f(t) = t^3 + 2t^2 + \frac{3}{2}t - \frac{3}{2}.$

41. Taking antiderivatives,
$f''(x) = 3\sin x + 4x^2$
$f'(x) = -3\cos x + \frac{4}{3}x^3 + c_1$
$f(x) = -3\sin x + \frac{1}{3}x^4 + c_1 x + c_2.$

42. Taking antiderivatives,
$f''(x) = x^{1/2} - 2\cos x$
$f'(x) = \frac{2}{3}x^{3/2} - 2\sin x + c_1$
$f(x) = \frac{4}{15}x^{5/2} + 2\cos x + c_1 x + c_2.$

43. Taking antiderivatives,
$f'''(x) = 4 - 2/x^3$
$f''(x) = 4x + x^{-2} + c_1$
$f'(x) = 2x^2 - x^{-1} + c_1 x + c_2$
$f(x) = \frac{2}{3}x^3 - \ln|x| + \frac{c_1}{2}x^2 + c_2 x + c_3$

44. Taking antiderivatives,
$f'''(x) = \sin x - e^x$
$f''(x) = -\cos x - e^x + c_1$
$f'(x) = -\sin x - e^x + c_1 x + c_2$
$f(x) = \cos x - e^x + \frac{c_1}{2}x^2 + c_2 x + c_3$

45. Position is the antiderivative of velocity,
$s(t) = 3t - 6t^2 + c.$
Since $s(0) = 3$, we have $c = 3$. Thus,
$s(t) = 3t - 6t^2 + 3.$

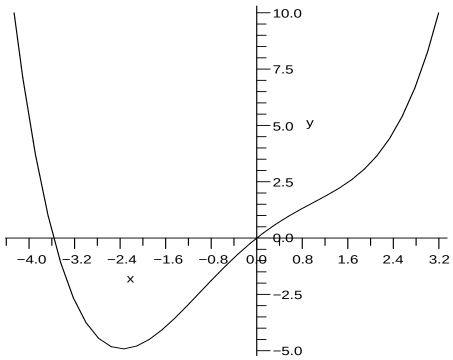
46. Position is the antiderivative of velocity,
$s(t) = -3e^{-t} - 2t + c.$
Since $s(0) = 0$, we have $-3 + c = 0$ and therefore $c = 3$. Thus,
$s(t) = -3e^{-t} - 2t + 3.$

47. First we find velocity, which is the antiderivative of acceleration,
$v(t) = -3\cos t + c_1.$
Since $v(0) = 0$ we have
$-3 + c_1 = 0$, $c_1 = 3$ and
$v(t) = -3\cos t + 3.$
Position is the antiderivative of velocity,
$s(t) = -3\sin t + 3t + c_2.$
Since $s(0) = 4$, we have $c_2 = 4$. Thus,
$s(t) = -3\sin t + 3t + 4.$

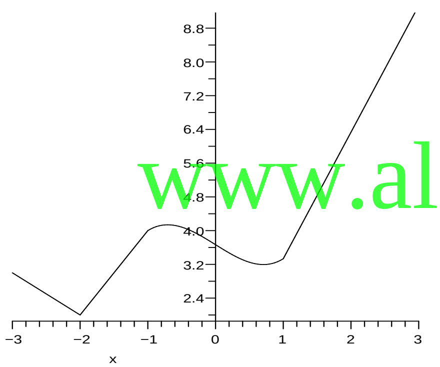
48. First we find velocity, which is the antiderivative of acceleration,
$v(t) = \frac{1}{3}t^3 + t + c_1.$
Since $v(0) = 4$ we have $c_1 = 4$ and
$v(t) = \frac{1}{3}t^3 + t + 4.$

Position is the antiderivative of velocity,
$s(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 4t + c_2.$
Since $s(0) = 0$, we have $c_2 = 0$. Thus,
$s(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 4t.$

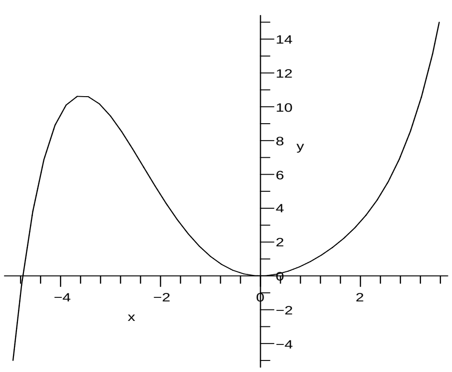
49. (a) There are many correct answers, but any correct answer will be a vertical shift of these answers.



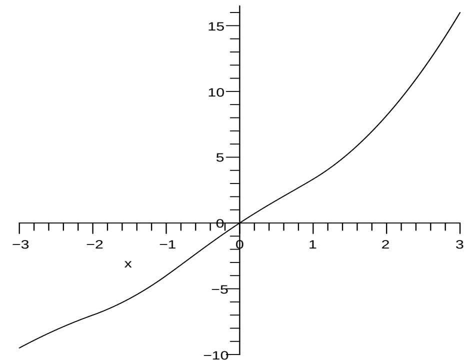
(b) There are many correct answers, but any correct answer will be a vertical shift of these answers.



50. (a) There are many correct answers, but any correct answer will be a vertical shift of these answers.



(b) There are many correct answers, but any correct answer will be a vertical shift of these answers.



51. We start by taking antiderivatives:
$f'(x) = x^2/2 - x + c_1$
$f(x) = x^3/6 - x^2/2 + c_1x + c_2$.
Now, we use the data that we are given. We know that $f(1) = 2$ and $f'(1) = 3$, which gives us
$3 = f'(1) = 1/2 - 1 + c_1$,
and
$1 = f(1) = 1/6 - 1/2 + c_1 + c_2$.
Therefore $c_1 = 7/2$ and $c_2 = -13/6$ and the function is
$f(x) = \dfrac{x^3}{6} - \dfrac{x^2}{2} + \dfrac{7x}{2} - \dfrac{13}{6}$.

52. We start by taking antiderivatives:
$f'(x) = 3x^2 + 4x + c_1$
$f(x) = x^3 + 2x^2 + c_1x + c_2$.
Now, we use the data that we are given. We know that $f(-1) = 1$ and $f'(-1) = 2$, which gives us
$2 = f'(-1) = -1 + c_1$,
and
$1 = f(-1) = 1 - c_1 + c_2$.
Therefore $c_1 = 3$ and $c_2 = 3$ and the function is
$f(x) = x^3 + 2x^2 + 3x - 3$.

53. $\dfrac{d}{dx}\left[\sin x^2\right] = 2x\cos x^2$
Therefore,
$\displaystyle\int 2x\cos x^2 dx = \sin x^2 + c$

54. $\dfrac{d}{dx}\left[(x^3+2)^{3/2}\right] = \dfrac{9}{2}x^2(x^3+2)^{1/2}$
Therefore,
$\displaystyle\int x^2\sqrt{x^3+2}dx = \frac{2}{9}(x^3+2)^{3/2} + c$

55. $\dfrac{d}{dx}\left[x^2\sin 2x\right] = 2(x\sin 2x + x^2\cos 2x)$
Therefore,

$$\int \left(x \sin 2x + x^2 \cos 2x\right) dx$$
$$= \frac{1}{2} x^2 \sin 2x + c$$

56. $\dfrac{d}{dx} \dfrac{x^2}{e^{3x}} = \dfrac{2xe^{3x} - 3x^2 e^{3x}}{e^{6x}}$
Therefore,
$$\int \frac{2xe^{3x} - 3x^2 e^{3x}}{e^{6x}} dx = \frac{x^2}{e^{3x}} + c$$

57. $\displaystyle\int \frac{x \cos(x^2)}{\sqrt{\sin(x^2)}} dx = \sqrt{\sin(x^2)} + c$

58. $\dfrac{d}{dx} \left(2\sqrt{x} \sin x\right) = 2\sqrt{x} \cos x + \dfrac{1}{\sqrt{x}} \sin x$
$$\int \left(2\sqrt{x} \cos x + \frac{1}{\sqrt{x}} \sin x\right) dx$$
$$= 2\sqrt{x} \sin x + c$$

59. Use a CAS to find antiderivatives and verify by computing the derivatives:
For 11.1(b):
$$\int \sec x dx = \ln|\sec x + \tan x| + c$$
Verify:
$$\frac{d}{dx} \ln|\sec x + \tan x|$$
$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$$
For 11.1(f):
$$\int x \sin 2x dx = \frac{\sin 2x}{4} - \frac{x \cos 2x}{2} + c$$
Verify:
$$\frac{d}{dx} \left(\frac{\sin 2x}{4} - \frac{x \cos 2x}{2}\right)$$
$$= \frac{2 \cos 2x}{4} - \frac{\cos 2x - 2x \sin 2x}{2}$$
$$= x \sin 2x$$

60. Use a CAS to find antiderivatives and verify by computing the derivatives:
For 31(a): The answer is too complicated to be presented here.
For 32(b): $\dfrac{1}{9}\left(3x + \sqrt{3} \ln \dfrac{2\sqrt{3} - 3x}{2\sqrt{3} + 3x}\right) + c$
Verify:
$$\frac{d}{dx}\left[\frac{1}{9}\left(3x + \sqrt{3} \ln \frac{2\sqrt{3} - 3x}{2\sqrt{3} + 3x}\right)\right]$$
$$= \frac{1}{9}\left(3 + \frac{2\sqrt{3} + 3x}{2\sqrt{3} - 3x} \cdot \frac{-3(2\sqrt{3} + 3x) - 3(2\sqrt{3} - 3x)}{(2\sqrt{3} + 3x)^2}\right)$$
$$= \frac{1}{9}\left(3 - \frac{36}{12 - 9x^2}\right) = \frac{x^2}{3x^2 - 4}$$
For 33(a): Almost the same as in Exercise 59, example 1.11 (b).
For 34(b): $\dfrac{1}{2} \ln \dfrac{x-1}{x+1} + c$
Verify:
$$\frac{d}{dx}\left(\frac{1}{2} \ln \frac{x-1}{x+1}\right)$$
$$= \frac{1}{2} \cdot \frac{x+1}{x-1} \cdot \frac{(x+1) - (x-1)}{(x+1)^2}$$
$$= \frac{1}{x^2 - 1}$$

61. Use a CAS to find antiderivatives and verify by computing the derivatives:

(a) $\displaystyle\int x^2 e^{-x^3} dx = -\frac{1}{3} e^{-x^3} + c$
Verify:
$$\frac{d}{dx}\left(-\frac{1}{3} e^{-x^3}\right)$$
$$= -\frac{1}{3} e^{-x^3} \cdot (-3x^2)$$
$$= x^2 e^{-x^3}$$

(b) $\displaystyle\int \frac{1}{x^2 - x} dx = \ln|x-1| - \ln|x| + c$ Verify:
$$\frac{d}{dx}(\ln|x-1| - \ln|x|)$$
$$= \frac{1}{x-1} - \frac{1}{x} = \frac{x - (x-1)}{x(x-1)}$$
$$= \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$$

(c) $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + c$
Verify:
$$\frac{d}{dx}\left[\ln|\sec x + \tan x|\right]$$
$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

62. Use a CAS to find antiderivatives and verify by computing the derivatives:

(a) $\displaystyle\int \frac{x}{x^4 + 1} dx = \frac{1}{2} \arctan x^2 + c$
Verify:
$$\frac{d}{dx}\left(\frac{1}{2} \arctan x^2\right)$$
$$= \frac{1}{2} \cdot \frac{1}{x^4 + 1} \cdot 2x = \frac{x}{x^4 + 1}$$

(b) $\displaystyle\int 3x \sin 2x dx$
$$= \frac{3}{4} \sin 2x - \frac{3x}{2} \cos 2x + c$$

Verify:

$$\frac{d}{dx}\left(\frac{3}{4}\sin 2x - \frac{3x}{2}\cos 2x\right)$$
$$= \frac{3}{2}\cos 2x - \frac{3}{2}\cos 2x + 3x\sin 2x$$
$$= 3x\sin 2x$$

(c) $\int \ln x dx = x\ln x - x + c$

Verify:

$$\frac{d}{dx}(x\ln x - x) = \ln x + 1 - 1$$
$$= \ln x$$

63. $\int \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1}(x) + c_1$

$\int \frac{-1}{\sqrt{1-x^2}}dx = -\sin^{-1}(x) + c_2$

Therefore,

$\cos^{-1} x + c_1 = -\sin^{-1} x + c_2$

Therefore,

$\sin^{-1} x + \cos^{-1} x =$ constant

To find the value of the constant, let x be any convenient value.

Suppose $x = 0$; then $\sin^{-1} 0 = 0$ and $\cos^{-1} 0 = \pi/2$, so

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

64. To derive these formulas, all that needs to be done is to take the derivatives to see that the integrals are correct:

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

65. To derive these formulas, all that needs to be done is to take the derivatives to see that the integrals are correct:

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(-e^{-x}) = e^{-x}$

66. (a) $\int \frac{1}{kx}dx = \frac{1}{k}\int \frac{1}{x}dx$

$= \frac{1}{k}\ln|x| + c_1$

(b) $\int \frac{1}{kx}dx = \frac{1}{k}\int \frac{k}{kx}dx$

$= \frac{1}{k}\ln|kx| + c_2$

Because

$\frac{1}{k}\ln|kx| = \frac{1}{k}(\ln|k| + \ln|x|)$

$= \frac{1}{k}\ln|x| + \frac{1}{k}\ln|k| = \frac{1}{k}\ln|x| + c$

The two antiderivatives are both correct.

67. The key is to find the velocity and position functions. We start with constant acceleration a, a constant. Then, $v(t) = at + v_0$ where v_0 is the initial velocity. The initial velocity is 30 miles per hour, but since our time is in seconds, it is probably best to work in feet per second (30mph = 44ft/s). $v(t) = at + 44$.

We know that the car accelerates to 50 mph (50mph = 73ft/s) in 4 seconds, so $v(4) = 73$.

Therefore, $a \cdot 4 + 44 = 73$ and $a = \frac{29}{4}$ ft/s

So,

$v(t) = \frac{29}{4}t + 44$ and

$s(t) = \frac{29}{8}t^2 + 44t + s_0$

where s_0 is the initial position. We can assume the the starting position is $s_0 = 0$.

Then, $s(t) = \frac{29}{8}t^2 + 44t$ and the distance traveled by the car during the 4 seconds is $s(4) = 234$ feet.

68. The key is to find the velocity and position functions. We start with constant acceleration a, a constant. Then, $v(t) = at + v_0$ where v_0 is the initial velocity. The initial velocity is 60 miles per hour, but since our time is in seconds, it is probably best to work in feet per second (60mph = 88ft/s). $v(t) = at + 88$.

We know that the car comes to rest in 3 seconds, so $v(3) = 0$.

Therefore,

$a(3) + 88 = 0$ and $a = -88/3$ft/s (the acceleration should be negative since the car is actually decelerating.

So,

$v(t) = -\frac{88}{3}t + 88$ and

$s(t) = -\frac{44}{3}t^2 + 88t + s_0$ where s_0 is the initial position. We can assume the the starting position is $s_0 = 0$.

Then, $s(t) = -\frac{44}{3}t^2 + 88t$ and the stopping distance is $s(3) = 132$ feet.

69. To estimate the acceleration over each interval, we estimate $v'(t)$ by computing the slope of the tangent lines. For example, for the interval $[0, 0.5]$:

$$a \approx \frac{v(0.5) - v(0)}{0.5 - 0} = -31.6 \text{ m/s}^2.$$

Notice, acceleration should be negative since the object is falling.

To estimate the distance traveled over the interval, we estimate the velocity and multiply by the time (distance is rate times time). For an estimate for the velocity, we will use the average of the velocities at the endpoints. For example, for the interval $[0, 0.5]$, the time interval is 0.5 and the velocity is -11.9. Therefore the position changed is $(-11.9)(0.5) = -5.95$ meters. The distance traveled will be 5.95 meters (distance should be positive).

Interval	Accel	Dist
$[0.0, 0.5]$	-31.6	5.95
$[0.5, 1.0]$	-2	12.925
$[1.0, 1.5]$	-11.6	17.4
$[1.5, 2.0]$	-3.6	19.3

70. To estimate the acceleration over each interval, we estimate $v'(t)$ by computing the slope of the tangent lines. For example, for the interval $[0, 1.0]$:
$a \approx \dfrac{v(1.0) - v(0)}{1.0 - 0} = -9.8 \text{ m/s}^2$.
Notice, acceleration should be negative since the object is falling.
To estimate the distance traveled over the interval, we estimate the velocity and multiply by the time (distance is rate times time). For an estimate for the velocity, we will use the average of the velocities at the endpoints. For example, for the interval $[0, 1.0]$, the time interval is 1.0 and the velocity is -4.9. Therefore the position changed is $(-4.9)(1.0) = -4.9$ meters. The distance traveled will be 4.9 meters (distance should be positive).

Interval	Accel	Dist
$[0.0, 1.0]$	-9.8	4.9
$[1.0, 2.0]$	-8.8	14.2
$[2.0, 3.0]$	-6.3	21.75
$[3.0, 4.0]$	-3.6	26.7

71. To estimate the speed over the interval, we first approximate the acceleration over the interval by averaging the acceleration at the endpoint of the interval. Then, the velocity will be the acceleration times the length of time. The slope of the tangent lines. For example, for the interval $[0, 0.5]$ the average acceleration is -0.9 and $v(0.5) = 70 + (-0.9)(0.5) = 69.55$.
And, the distance traveled is the speed times the length of time. For the time $t = 0.5$, the distance would be $\dfrac{70 + 69.55}{2} \times 0.5 \approx 34.89$ meters.

Time	Speed	Dist
0	70	0
0.5	69.55	34.89
1.0	70.3	69.85
1.5	70.35	105.01
2.0	70.65	104.26

72. To estimate the speed over the interval, we first approximate the acceleration over the interval by averaging the acceleration at the endpoint of the interval. Then, the velocity will be the acceleration times the length of time. the slope of the tangent lines. For example, for the interval $[0.0, 0.5]$ the average acceleration is -0.8 and $v(0.5) = 20 + (-0.8)(.5) = 19.6$. Of course, speed is the absolute value of the velocity.
And, the distance traveled is the average speed times the length of time. For the time $t = 0.5$, the distance would be $\dfrac{20 + 19.6}{2} \times 0.5 = 9.9$ meters.

Time	Speed	Dist
0	20	0
0.5	19.6	9.9
1.0	17.925	19.281
1.5	16.5	27.888
2.0	16.125	34.044

4.2 Sums And Sigma Notation

1. The given sum is the sum of twice the squares of the integers from 1 to 14.
$$2(1)^2 + 2(2)^2 + 2(3)^2 + \ldots + 2(14)^2 = \sum_{i=1}^{14} 2i^2$$

2. The given sum is the sum of squares roots of the integers from 1 to 14.
$$\sqrt{2-1} + \sqrt{3-1} + \sqrt{4-1} + \ldots + \sqrt{15-1}$$
$$= \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{13} + \sqrt{14}$$
$$= \sum_{i=1}^{14} \sqrt{i}$$

3. (a) $\displaystyle\sum_{i=1}^{50} i^2 = \frac{(50)(51)(101)}{6} = 42,925$

(b) $\displaystyle\left(\sum_{i=1}^{50} i\right)^2 = \left(\frac{50(51)}{2}\right)^2 = 1,625,625$

4. (a) $\displaystyle\sum_{i=1}^{10} \sqrt{i}$
$= 1 + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$

$$+\sqrt{7}+\sqrt{8}+\sqrt{9}+\sqrt{10}$$
$$\approx 22.47$$

(b) $\sqrt{\sum_{i=1}^{10} i} = \sqrt{\frac{10(11)}{2}} = \sqrt{55}$

5. $\sum_{i=1}^{6} 3i^2 = 3+12+27+48+75+108$
$= 273$

6. $\sum_{i=3}^{7} i^2 + i = 12+20+30+42+56$
$= 160$

7. $\sum_{i=6}^{10}(4i+2)$
$= (4(6)+2)+(4(7)+2)+(4(8)+2)$
$+ (4(9)+2)+(4(10)+2)$
$= 26+30+34+38+42$
$= 170$

8. $\sum_{i=6}^{8}(i^2+2)$
$= (6^2+2)+(7^2+2)+(8^2+2)$
$= 38+51+66 = 155$

9. $\sum_{i=1}^{70}(3i-1) = 3\cdot\sum_{i=1}^{70} i - 70$
$= 3\cdot\frac{70(71)}{2} - 70 = 7,385$

10. $\sum_{i=1}^{45}(3i-4) = 3\sum_{i=1}^{45} i - 4\sum_{i=1}^{45} 1$
$= 3\left(\frac{45(46)}{2}\right) - 4(45) = 2925$

11. $\sum_{i=1}^{40}(4-i^2) = 160 - \sum_{i=1}^{40} i^2$
$= 160 - \frac{(40)(41)(81)}{6}$
$= 160 - 22,140 = -21,980$

12. $\sum_{i=1}^{50}(8-i) = 8\sum_{i=1}^{50} 1 - \sum_{i=1}^{50} i$
$= 8(50) - \frac{50(51)}{2} = -875$

13. $\sum_{n=1}^{100}\left(n^2-3n+2\right)$
$= \sum_{n=1}^{100} n^2 - 3\sum_{n=1}^{100} n + \sum_{n=1}^{100} 2$
$= \frac{(100)(101)(201)}{6} - 3\frac{100(101)}{2} + 200$
$= 338,350 - 15,150 + 200 = 323,400$

14. $\sum_{n=1}^{140}\left(n^2+2n-4\right)$
$= \sum_{n=1}^{140} n^2 + 2\sum_{n=1}^{140} n - \sum_{n=1}^{140} 4$
$= \frac{(140)(141)(281)}{6} + 2\left(\frac{140(141)}{2}\right) - 4(140)$
$= 943,670$

15. $\sum_{i=3}^{30}\left[(i-3)^2 + i - 3\right]$
$= \sum_{i=3}^{30}(i-3)^2 + \sum_{i=3}^{30}(i-3)$
$= \sum_{n=0}^{27} n^2 + \sum_{n=0}^{27} n$ (substitute $i-3=n$)
$= 0 + \sum_{n=1}^{27} n^2 + 0 + \sum_{n=1}^{27} n$
$= \frac{27(28)(55)}{6} + \frac{27(28)}{2} = 7308$

16. $\sum_{i=4}^{20}(i-3)(i+3) = \sum_{i=4}^{20}\left(i^2-9\right)$
$= \sum_{i=4}^{20} i^2 - 9\sum_{i=4}^{20} 1$
$= \sum_{i=1}^{20} i^2 - \sum_{i=1}^{3} i^2 - 9\sum_{i=4}^{20} 1$
$= \frac{20(21)(41)}{6} - 1 - 4 - 9 - 9(17)$
$= 2703$

17. $\sum_{k=3}^{n}\left(k^2-3\right)$
$= \sum_{k=3}^{n} k^2 + \sum_{k=3}^{n}(-3)$
$= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{2} k^2$
$+ \sum_{k=1}^{n}(-3) - \sum_{k=1}^{2}(-3)$
$= \frac{n(n+1)(2n+1)}{6} - 1 - 4$
$+ (-3)n - (-3)(2)$
$= \frac{n(n+1)(2n+1)}{6} - 5 - 3n + 6$

$$= \frac{n\,(n+1)\,(2n+1)}{6} - 3n + 1$$

18. $\displaystyle\sum_{k=0}^{n} \left(k^2 + 5\right)$

$$= \sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} 5$$
$$= 0 + \sum_{k=1}^{n} k^2 + 5 + \sum_{k=1}^{n} 5$$
$$= \frac{n\,(n+1)\,(2n+1)}{6} + 5 + 5n$$

19. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{5} (x_i^2 + 4x_i) \cdot 0.2$$
$$= (0.2^2 + 4(0.2))(0.2) + \ldots + (1^2 + 4)(0.2)$$
$$= (0.84)(0.2) + (1.76)(0.2) + (2.76)(0.2) + (3.84)(0.2) + (5)(0.2)$$
$$= 2.84$$

20. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{5} (3x_i + 5) \cdot 0.4$$
$$= (3(0.4) + 5)(0.4) + \ldots + (3(2) + 5)(0.4)$$
$$= (6.2)(0.4) + (7.4)(0.4) + (8.6)(0.4) + (9.8)(0.4) + (11)(0.4)$$
$$= 17.2$$

21. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{10} (4x_i^2 - 2) \cdot 0.1$$
$$= (4(2.1)^2 - 2)(0.1) + \ldots + (4(3)^2 - 2)(0.1)$$
$$= (15.64)(0.1) + (17.36)(0.1) + (19.16)(0.1) + (21.04)(0.1) + (23)(0.1) + (25.04)(0.1) + (27.16)(0.1) + (29.36)(0.1) + (31.64)(0.1) + (34)(0.1)$$
$$= 24.34$$

22. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{10} (x^3 + 4) \cdot 0.1$$
$$= ((2.05)^3 + 4)(0.1) + \ldots + ((2.95)^3 + 4)(0.1)$$
$$= (202.4375)(0.1)$$
$$= 20.24375$$

23. $\displaystyle\sum_{i=1}^{n} \frac{1}{n} \left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right) \right]$

$$= \frac{1}{n} \left[\sum_{i=1}^{n} \frac{i^2}{n^2} + 2 \sum_{i=1}^{n} \frac{i}{n} \right]$$
$$= \frac{1}{n} \left[\frac{1}{n^2} \sum_{i=1}^{n} i^2 + \frac{2}{n} \sum_{i=1}^{n} i \right]$$
$$= \frac{1}{n} \left[\frac{1}{n^2} \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{2}{n} \left(\frac{n(n+1)}{2} \right) \right]$$
$$= \frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{n^2}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right) \right]$$
$$= \lim_{n\to\infty} \left[\frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{n^2} \right]$$
$$= \frac{2}{6} + 1 = \frac{4}{3}$$

24. $\displaystyle\sum_{i=1}^{n} \frac{1}{n} \left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right) \right]$

$$= \frac{1}{n} \left[\sum_{i=1}^{n} \frac{i^2}{n^2} - 5 \sum_{i=1}^{n} \frac{i}{n} \right]$$
$$= \frac{1}{n} \left[\frac{1}{n^2} \sum_{i=1}^{n} i^2 - \frac{5}{n} \sum_{i=1}^{n} i \right]$$
$$= \frac{1}{n} \left[\frac{1}{n^2} \left(\frac{n(n+1)(2n+1)}{6} \right) - \frac{5}{n} \left(\frac{n(n+1)}{2} \right) \right]$$
$$= \frac{n(n+1)(2n+1)}{6n^3} - \frac{5n(n+1)}{2n^2}$$
$$= \frac{-13n^2 - 12n + 1}{6n^2}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right) \right]$$
$$= \lim_{n\to\infty} \frac{-13n^2 - 12n + 1}{6n^2}$$
$$= \lim_{n\to\infty} -\frac{13}{6} - \frac{12}{6n} + \frac{1}{6n^2}$$

$$= -\frac{13}{6}$$

25. $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$

$$= \frac{1}{n}\left[16\sum_{i=1}^{n}\frac{i^2}{n^2} - 2\sum_{i=1}^{n}\frac{i}{n}\right]$$

$$= \frac{1}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$

$$= \frac{1}{n}\left[\frac{16}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{2}{n}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \frac{16n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{16n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^2}\right]$$

$$= \frac{16}{3} - 1 = \frac{13}{3}$$

26. $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}\frac{4i^2}{n^2} + 4\sum_{i=1}^{n}\frac{i}{n}\right]$$

$$= \frac{1}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 + \frac{4}{n}\sum_{i=1}^{n} i\right]$$

$$= \frac{1}{n}\left[\frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{4}{n}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \frac{4n(n+1)(2n+1)}{6n^3} + \frac{4n(n+1)}{2n^2}$$

$$= \frac{10n^2+12n+2}{3n^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{10n^2+12n+2}{3n^2}$$

$$= \lim_{n\to\infty}\frac{10}{3} + \frac{12}{3n} + \frac{2}{3n^2} = \frac{10}{3}$$

27. Want to prove that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

is true for all integers $n \ge 1$.

For $n = 1$, we have

$$\sum_{i=1}^{1} i^3 = 1 = \frac{1^2(1+1)^2}{4},$$

as desired.

So the proposition is true for $n = 1$.

Next, assume that

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4},$$

for some integer $k \ge 1$.

In this case, we have by the induction assumption that for $n = k+1$,

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{n^2(n+1)^2}{4}$$

as desired.

28. Want to prove that

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

is true for all integers $n \ge 1$.

For $n = 1$, we have

$$\sum_{i=1}^{1} i^3 = 1 = \frac{1^2(1+1)^2(2+2-1)}{12},$$

as desired.

So the proposition is true for $n = 1$.

Next, assume that

$$\sum_{i=1}^{k} i^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12},$$

for some integer $k \ge 1$.

In this case, we have by the induction assumption that for $n = k+1$,

$$\sum_{i=1}^{n} i^5 = \sum_{i=1}^{k+1} i^5 = \sum_{i=1}^{k} i^5 + (k+1)^5$$

$$= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + (k+1)^5$$

$$= \frac{k^2(k+1)^2(2k^2+2k-1) + 12(k+1)^5}{12}$$

$$= \frac{(k+1)^2[k^2(2k^2+2k-1) + 12(k+1)^3]}{12}$$

$= \dfrac{(k+1)^2[2k^4+14k^3+35k^2+36k+12]}{12}$

$= \dfrac{(k+1)^2(k^2+4k+4)(2k^2+6k+3)}{12}$

$= \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$

as desired.

29. $\displaystyle\sum_{i=1}^{10}(i^3-3i+1)$

$= \displaystyle\sum_{i=1}^{10} i^3 - 3\sum_{i=1}^{10} i + 10$

$= \dfrac{100(11)^2}{4} - 3\dfrac{10(11)}{2} + 10$

$= 2,870$

30. $\displaystyle\sum_{i=1}^{20}(i^3+2i)$

$= \displaystyle\sum_{i=1}^{20} i^3 + 2\sum_{i=1}^{20} i$

$= \dfrac{400(21)^2}{4} + 2\dfrac{20(21)}{2} = 44,520$

31. $\displaystyle\sum_{i=1}^{100}(i^5-2i^2)$

$= \displaystyle\sum_{i=1}^{100} i^5 - 2\sum_{i=1}^{100} i^2$

$= \dfrac{(100^2)(101^2)[2(100^2)+2(100)-1]}{12}$

$- 2\dfrac{100(101)(201)}{6}$

$= 171,707,655,800$

32. $\displaystyle\sum_{i=1}^{100}(2i^5+2i+1)$

$= 2\displaystyle\sum_{i=1}^{100} i^5 + 2\sum_{i=1}^{100} i + 100$

$= 2\dfrac{(100^2)(101^2)[2(100^2)+2(100)-1]}{12}$

$+ 2\cdot\dfrac{100(101)}{2} + 100$

$= 343,416,675,200$

33. $\displaystyle\sum_{i=1}^{n}(ca_i+db_i) = \sum_{i=1}^{n} ca_i + \sum_{i=1}^{n} db_i$

$= c\displaystyle\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$

34. When $n=0$, $a = \dfrac{a-ar}{1-r}$.

Assume the formula holds for $n=k-1$, which gives

$a + ar + \cdots ar^{k-1} = \dfrac{a-ar^k}{1-r}$.

Then for $n=k$,

we have $a + ar + \cdots ar^k$

$= a + ar + \cdots ar^{k-1} + ar^k$

$= \dfrac{a-ar^k}{1-r} + ar^k$

$= \dfrac{a-ar^k+ar^k(1-r)}{1-r}$

$= \dfrac{a-ar^k+ar^k-ar^{k+1}}{1-r}$

$= \dfrac{a-ar^{k+1}}{1-r}$

$= \dfrac{a-ar^{n+1}}{1-r}$

as desired.

35. $\displaystyle\sum_{i=1}^{n} e^{6i/n}\left(\frac{6}{n}\right)$

$= \dfrac{6}{n}\displaystyle\sum_{i=1}^{n} e^{6i/n}$

$= \dfrac{6}{n}\left(\dfrac{e^{6/n}-e^6}{1-e^{6/n}}\right)$

$= \dfrac{6}{n}\left(\dfrac{1-e^6}{1-e^{6/n}} - 1\right)$

$= \dfrac{6}{n}\dfrac{1-e^6}{1-e^{6/n}} - \dfrac{6}{n}$

Now $\displaystyle\lim_{x\to\infty}\frac{6}{n} = 0$, and

$\displaystyle\lim_{x\to\infty}\frac{6}{n}\frac{1-e^6}{1-e^{6/n}}$

$= 6(1-e^6)\displaystyle\lim_{x\to\infty}\frac{1/n}{1-e^{6/n}}$

$= 6(1-e^6)\displaystyle\lim_{x\to\infty}\frac{1}{-6e^{6/n}}$

$= e^6 - 1.$

Thus $\displaystyle\lim_{x\to\infty}\sum_{i=1}^{n} e^{6i/n}\frac{6}{n} = e^6 - 1.$

36. $\displaystyle\sum_{i=1}^{n} e^{(2i)/n}\frac{2}{n}$

$= \dfrac{2}{n}\left(\dfrac{e^{2/n}-e^2}{1-e^{2/n}}\right)$

$= \dfrac{2}{n}\left(\dfrac{1-e^2}{1-e^{2/n}} - 1\right)$

$= \frac{2}{n}\frac{1-e^2}{1-e^{2/n}} - \frac{2}{n}$

Now $\lim_{x\to\infty} \frac{2}{n} = 0$, and

$\lim_{x\to\infty} \frac{2}{n}\frac{1-e^2}{1-e^{2/n}}$

$= 2(1-e^2)\lim_{x\to\infty} \frac{1/n}{1-e^{2/n}}$

$= 2(1-e^2)\lim_{x\to\infty} \frac{1}{-2e^{2/n}}$

$= e^2 - 1.$

Thus $\lim_{x\to\infty} \sum_{i=1}^{n} e^{2i/n}\frac{2}{n} = e^2 - 1.$

37. Distance
$= 50(2) + 60(1) + 70(1/2) + 60(3)$
$= 375$ miles.

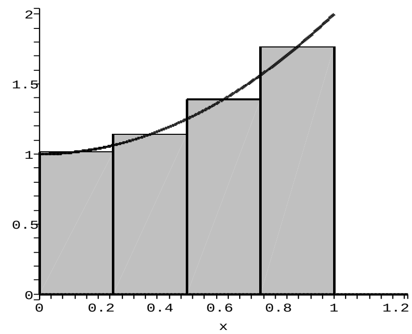
38. Distance
$= 50(1) + 40(1) + 60(1/2) + 55(3)$
$= 285$ miles.

39. On the time interval $[0, 0.25]$, the estimated velocity is the average velocity $\frac{120+116}{2} = 118$ feet per second.
We estimate the distance traveled during the time interval $[0, 0.25]$ to be
$(118)(0.25 - 0) = 29.5$ feet.
Altogether, the distance traveled is estimated as
$= (236/2)(0.25) + (229/2)(0.25)$
$+ (223/2)(0.25) + (218/2)(0.25)$
$+ (214/2)(0.25) + (210/2)(0.25)$
$+ (207/2)(0.25) + (205/2)(0.25)$
$= 217.75$ feet.

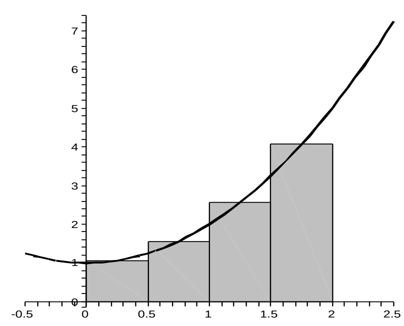
40. On the time interval $[0, 0.5]$, the estimated velocity is the average velocity $\frac{10+14.9}{2} = 12.45$ meters per second. We estimate the distance fallen during the time interval $[0, 0.5]$ to be
$(12.45)(0.5 - 0) = 6.225$ meters.
Altogether, the distance fallen (estimated)
$= (12.45)(0.5) + (17.35)(0.5)$
$+ (22.25)(0.5) + (27.15)(0.5)$
$+ (32.05)(0.5) + (36.95)(0.5)$
$+ (41.85)(0.5) + (46.75)(0.5)$
$= 118.4$ meters.

4.3 Area

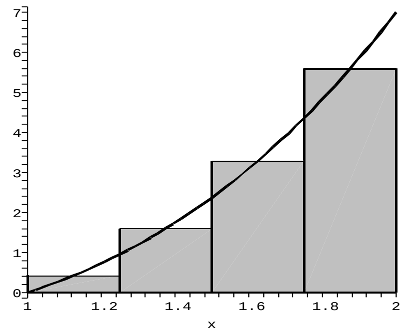
1. (a) Evaluation points:
0.125, 0.375, 0.625, 0.875.
Notice that $\Delta x = 0.25$.
$A_4 = [f(0.125) + f(0.375) + f(0.625) + f(0.875)](0.25)$
$= [(0.125)^2 + 1 + (0.375)^2 + 1 + (0.625)^2 + 1 + (0.875)^2 + 1](0.25)$
$= 1.38125.$



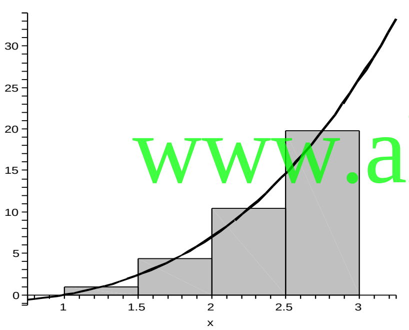
(b) Evaluation points:
0.25, 0.75, 1.25, 1.75.
Notice that $\Delta x = 0.5$.
$A_4 = [f(0.25) + f(0.75) + f(1.25) + f(1.75)](0.5)$
$= [(0.25)^2 + 1 + (0.75)^2 + 1 + (1.25)^2 + 1 + (1.75)^2 + 1](0.5)$
$= 4.625.$



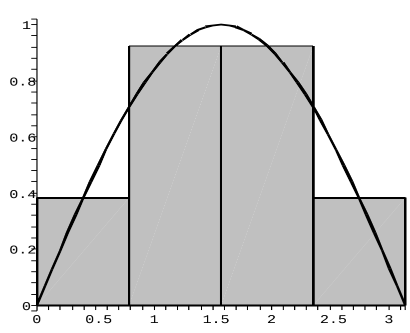
2. (a) Evaluation points:
1.125, 1.375, 1.625, 1.875.
Notice that $\Delta x = 0.25$.
$A_4 = [f(1.125) + f(1.375) + f(1.625) + f(1.875)](0.25)$
$= [(1.125)^3 - 1 + (1.375)^3 - 1 + (1.625)^3 - 1 + (1.875)^3 - 1](0.25)$
$= 2.7265625.$



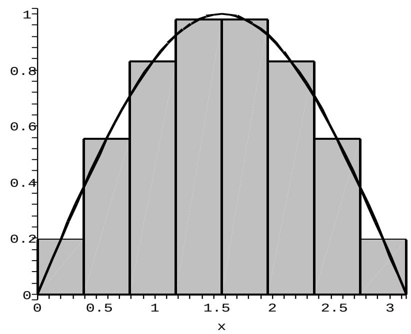
(b) Evaluation points:
1.25, 1.75, 2.25, 2.75.
Notice that $\Delta x = 0.5$.
$A_4 = [f(1.25) + f(1.75) + f(2.25) + f(2.75)](0.5)$
$= [(1.25)^3 - 1 + (1.75)^3 - 1 + (2.25)^3 - 1 + (2.75)^3 - 1](0.5)$
$= 17.75.$



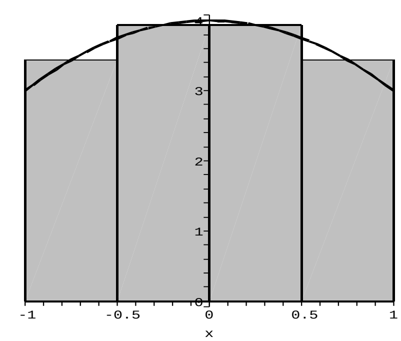
3. (a) Evaluation points:
$\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$.
Notice that $\Delta x = \pi/4$.
$A_4 = [f(\pi/8) + f(3\pi/8) + f(5\pi/8) + f(7\pi/8)](\pi/4)$
$= [\sin(\pi/8) + \sin(3\pi/8) + \sin(5\pi/8) + \sin(7\pi/8)](\pi/4)$
$= 2.05234.$



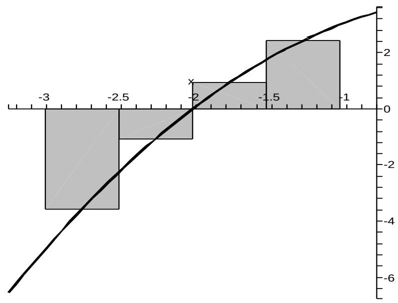
(b) Evaluation points:
$\pi/16$, $3\pi/16$, $5\pi/16$, $7\pi/16$, $9\pi/16$, $11\pi/16$, $13\pi/16$, $15\pi/16$.
Notice that $\Delta x = \pi/8$.
$A_4 = [f(\pi/16) + f(3\pi/16) + f(5\pi/16) + f(7\pi/16) + f(9\pi/16) + f(11\pi/16) + f(13\pi/16) + f(15\pi/16)](\pi/8)$
$= [\sin(\pi/16) + \sin(3\pi/16) + \sin(5\pi/16) + \sin(7\pi/16) + \sin(9\pi/16) + \sin(11\pi/16) + \sin(13\pi/16) + \sin(15\pi/16)](\pi/8)$
$= 2.0129.$



4. (a) Evaluation points:
-0.75, -0.25, 0.25, 0.75.
Notice that $\Delta x = 0.5$.
$A_4 = [f(-0.75) + f(-0.25) + f(0.25) + f(0.75)](0.5)$
$= [4 - (-0.75)^2 + 4 - (-0.25)^2 + 4 - (0.25)^2 + 4 - (0.75)^2](0.5)$
$= 7.375.$



(b) Evaluation points:
-2.75, -2.25, -1.75, -1.25.
Notice that $\Delta x = 0.5$.
$A_4 = [f(-2.75) + f(-2.25) + f(-1.75) + f(-1.25)](0.5)$
$= [4 - (-2.75)^2 + 4 - (-2.25)^2 + 4 - (-1.75)^2 + 4 - (-1.25)^2](0.5)$
$= -0.625.$



5. (a) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{16} \sum_{i=0}^{15} \left[\left(\frac{i}{16} \right)^2 + 1 \right] \approx 1.3027$$

(b) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \dfrac{\Delta x}{2}$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{16} \sum_{i=0}^{15} \left[\left(\frac{i}{16} + \frac{1}{32} \right)^2 + 1 \right]$$
$$\approx 1.3330$$

(c) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{16} \sum_{i=0}^{15} \left[\left(\frac{i}{16} + \frac{1}{16} \right)^2 + 1 \right]$$
$$\approx 1.3652$$

6. (a) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{8} \sum_{i=0}^{15} \left[\left(\frac{i}{8} \right)^2 + 1 \right] \approx 4.4219$$

(b) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \dfrac{\Delta x}{2}$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{8} \sum_{i=0}^{15} \left[\left(\frac{i}{8} + \frac{1}{16} \right)^2 + 1 \right] \approx 4.6640$$

(c) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{8} \sum_{i=0}^{15} \left[\left(\frac{i}{8} + \frac{1}{8} \right)^2 + 1 \right] \approx 4.9219$$

7. (a) There are 16 rectangles and the evaluation points are the left endpoints which are given by
$c_i = 1 + i\Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{3}{16} \sum_{i=0}^{15} \sqrt{1 + \frac{3i}{16} + 2} \approx 6.2663$$

(b) There are 16 rectangles and the evaluation points are the midpoints which are given by
$c_i = 1 + i\Delta x + \dfrac{\Delta x}{2}$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{3}{16} \sum_{i=0}^{15} \sqrt{1 + \frac{3i}{16} + \frac{3}{32} + 2}$$
$$\approx 6.3340$$

(c) There are 16 rectangles and the evaluation points are the right endpoints which are given by
$c_i = 1 + i\Delta x$ where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$
$$= \frac{3}{16} \sum_{i=1}^{16} \sqrt{1 + \frac{3i}{16} + 2} \approx 6.4009$$

8. (a) There are 16 rectangles and the evaluation points are the left endpoints which are given by
$c_i = -1 + i\Delta x - \Delta x$
where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$

$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8}-\frac{1}{8})} \approx 4.0991$

(b) There are 16 rectangles and the evaluation points are the midpoints which are given by
$c_i = -1 + i\Delta x - \frac{\Delta x}{2}$
where i is from 1 to 16.
$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$
$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8}-\frac{1}{16})} \approx 3.6174$

(c) There are 16 rectangles and the evaluation points are the right endpoints which are given by
$c_i = -1 + i\Delta x$ where i is from 1 to 16.
$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$
$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8})} \approx 3.1924$

9. (a) There are 50 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 49.
$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$
$= \frac{\pi}{100}\sum_{i=0}^{50} \cos\left(\frac{\pi i}{100}\right) \approx 1.0156$

(b) There are 50 rectangles and the evaluation points are given by $c_i = \frac{\Delta x}{2} + i\Delta x$ where i is from 0 to 49.
$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$
$= \frac{\pi}{100}\sum_{i=0}^{50} \cos\left(\frac{\pi}{200} + \frac{\pi i}{100}\right)$
≈ 1.00004

(c) There are 50 rectangles and the evaluation points are given by $c_i = \Delta x + i\Delta x$ where i is from 0 to 49.
$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$
$= \frac{\pi}{100}\sum_{i=0}^{50} \cos\left(\frac{\pi}{100} + \frac{\pi i}{100}\right)$
≈ 0.9842

10. (a) There are 100 rectangles and the evaluation points are left endpoints which are given by $c_i = -1 + i\Delta x - \Delta x$ where i is from 1 to 100.
$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$
$= \frac{2}{100}\sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100} - \frac{2}{100}\right)^3 - 1\right]$
≈ -2.02

(b) There are 100 rectangles and the evaluation points are midpoints which are given by $c_i = -1 + i\Delta x - \frac{\Delta x}{2}$ where i is from 1 to 100.
$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$
$= \frac{2}{100}\sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100} - \frac{1}{100}\right)^3 - 1\right]$
$= -2$

(c) There are 100 rectangles and the evaluation points are right endpoints which are given by $c_i = -1 + i\Delta x$ where i is from 1 to 100.
$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$
$= \frac{2}{100}\sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100}\right)^3 - 1\right] \approx -1.98$

11. (a) $\Delta x = \frac{1}{n}$. We will use right endpoints as evaluation points, $x_i = \frac{i}{n}$.
$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$
$= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] = \frac{1}{n^3}\sum_{i=1}^{n} i^2 + 1$
$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + 1$
$= \frac{8n^2+3n+1}{6n^2}$
Now to compute the exact area, we take the limit as $n \to \infty$:
$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{8n^2+3n+1}{6n^2}$
$= \lim_{n\to\infty} \frac{8}{6} + \frac{3}{6n} + \frac{1}{6n^2} = \frac{4}{3}$

(b) $\Delta x = \frac{2}{n}$. We will use right endpoints as evaluation points, $x_i = \frac{2i}{n}$.
$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+1\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+1\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2+\frac{2}{n}\sum_{i=1}^{n}1$$
$$= \frac{8}{n^3}\sum_{i=1}^{n}i^2+2$$
$$= \frac{8}{n^3}\left[\frac{n\,(n+1)\,(2n+1)}{6}\right]+2$$
$$= \frac{8}{n^2}\left[\frac{(n+1)\,(2n+1)}{6}\right]+2$$
$$= \frac{4}{3n^2}\left(2n^2+3n+1\right)+2$$
$$= \frac{14n^2+12n+4}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\frac{14n^2+12n+4}{3n^2}$$
$$= \frac{14}{3}$$

(c) $\Delta x = \frac{2}{n}$ We will use right endpoints as evaluation points,$x_i = 1+\frac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f\,(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n}\left({x_i}^2+1\right)\left(\frac{2}{n}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(\left(1+\frac{2i}{n}\right)^2+1\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(2+\frac{4i}{n}+\frac{4i^2}{n^2}\right)$$
$$= 4+\frac{8}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2$$
$$= 4+\frac{8}{n^2}\left(\frac{n\,(n+1)}{2}\right)$$
$$+\frac{8}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right)$$
$$= 4+\left(\frac{4n+4}{n}\right)+\left[\frac{8n^2+12n+4}{3n^2}\right]$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(4+\frac{4n+4}{n}+\frac{8n^2+12n+4}{3n^2}\right)$$
$$= 4+4+\frac{8}{3} = \frac{32}{3}$$

12. (a) $\Delta x = \frac{1}{n}$. We will use right endpoints as evaluation points, $x_i = \frac{i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2+3\left(\frac{i}{n}\right)\right]$$
$$= \frac{1}{n^3}\sum_{i=1}^{n}i^2+\frac{3}{n^2}\sum_{i=1}^{n}i$$
$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$+\frac{3}{n^2}\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{11n^2+12n+1}{6n^2}$$

Now to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\frac{11n^2+12n+1}{6n^2}$$
$$= \lim_{n\to\infty}\frac{11}{6}+\frac{12}{6n}+\frac{1}{6n^2} = \frac{11}{6}$$

(b) $\Delta x = \frac{2}{n}$. We will use right endpoints as evalution points, $x_i = \frac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+3\left(\frac{2i}{n}\right)\right]$$
$$= \frac{8}{n^3}\sum_{i=1}^{n}i^2+\frac{12}{n^2}\sum_{i=1}^{n}i$$
$$= \frac{8}{n^3}\left[\frac{n\,(n+1)\,(2n+1)}{6}\right]$$
$$+\frac{12}{n^2}\left[\frac{n\,(n+1)}{2}\right]$$
$$= \left[\frac{\left(8n^2+12n+4\right)}{3n^2}\right]+\left[\frac{6n+6}{n}\right]$$

Now, to compute the exact area, we take the limit as $n \to \infty$: $A = \lim_{n\to\infty} A_n$

$$= \lim_{n\to\infty}\left(\frac{\left(8n^2+12n+4\right)}{3n^2}+\frac{6n+6}{n}\right)$$
$$= \frac{8}{3}+6 = \frac{26}{3}$$

(c) $\Delta x = \frac{2}{n}$. We will use right endpoints as evalution points, $x_i = 1 + \frac{2i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} \left[{x_i}^2 + 3x_i\right] \frac{2}{n} \\
&= \frac{2}{n} \sum_{i=1}^{n} \left[\left(1 + \frac{2i}{n}\right)^2 + 3\left(1 + \frac{2i}{n}\right)\right] \\
&= \frac{2}{n} \sum_{i=1}^{n} \left(4 + \frac{10i}{n} + \frac{4i^2}{n^2}\right) \\
&= 8 + \frac{20}{n^2} \sum_{i=1}^{n} i + \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\
&= 8 + \frac{20}{n^2}\left(\frac{n(n+1)}{2}\right) \\
&\quad + \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 8 + \frac{10}{n}(n+1) + \frac{4}{3n^2}\left(2n^2 + 3n + 1\right)
\end{aligned}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \\
&\left[8 + \frac{10}{n}(n+1) + \frac{4}{3n^2}\left(2n^2 + 3n + 1\right)\right] \\
&= 8 + 10 + \frac{8}{3} = \frac{62}{3}
\end{aligned}$$

13. (a) $\Delta x = \frac{1}{n}$. We will use right endpoints as evalution points, $x_i = \frac{i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \frac{1}{n} \sum_{i=1}^{n} \left[2\left(\frac{i}{n}\right)^2 + 1\right] \\
&= \frac{2}{n^3} \sum_{i=1}^{n} i^2 + 1 \\
&= \frac{2}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + 1 \\
&= \frac{\left(5n^2 + n + 1\right)}{3n^2}
\end{aligned}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \left[\frac{\left(5n^2 + n + 1\right)}{3n^2}\right] = \frac{5}{3}.
\end{aligned}$$

(b) $\Delta x = \frac{2}{n}$. We will use right endpoints as evalution points, $x_i = -1 + \frac{2i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} \left(2{x_i}^2 + 1\right)\left(\frac{2}{n}\right) \\
&= \frac{2}{n} \sum_{i=1}^{n} \left(2\left(-1 + \frac{2i}{n}\right)^2 + 1\right) \\
&= \frac{2}{n} \sum_{i=1}^{n} \left(3 - \frac{8i}{n} + \frac{8i^2}{n^2}\right) \\
&= 6 - \frac{16}{n^2} \sum_{i=1}^{n} i + \frac{16}{n^3} \sum_{i=1}^{n} i^2 \\
&= 6 - \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) \\
&\quad + \frac{16}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 6 - \left(\frac{8n+8}{n}\right) + \left(\frac{16n^2 + 24n + 8}{3n^2}\right)
\end{aligned}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \\
&\left[6 - \left(\frac{8n+8}{n}\right) + \left(\frac{16n^2 + 24n + 8}{3n^2}\right)\right] \\
&= 6 - 8 + \frac{16}{3} = \frac{10}{3}
\end{aligned}$$

(c) $\Delta x = \frac{2}{n}$. We will use right endpoints as evaluation points, $x_i = 1 + \frac{2i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \frac{2}{n} \sum_{i=1}^{n} 2\left(1 + \frac{2i}{n}\right)^2 + 1 \\
&= \frac{2}{n} \sum_{i=1}^{n} \left(\frac{8i^2}{n^2} + \frac{8i}{n} + 3\right) \\
&= \frac{16}{n^3} \sum_{i=1}^{n} i^2 + \frac{16}{n^2} \sum_{i=1}^{n} i + 6 \\
&= \frac{16}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&\quad + \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) + 6 \\
&= \frac{16n(n+1)(2n+1)}{6n^3}
\end{aligned}$$

$$+\frac{16n(n+1)}{2n^2}+6$$

Now to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{16n(n+1)(2n+1)}{6n^3} + \frac{16n(n+1)}{2n^2}+6\right)$$
$$= \lim_{n\to\infty} \frac{32}{6}+\frac{16}{2}+6 \quad = \frac{58}{3}$$

14. (a) $\Delta x = \frac{1}{n}$. We will use right endpoints as evalution points, $x_i = \frac{i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(4{x_i}^2 - x_i\right)\frac{1}{n}$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(\frac{i}{n}\right)^2 - \left(\frac{i}{n}\right)\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{4i^2}{n^2}-\frac{i}{n}\right)\right]$$
$$= \frac{4}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}$$
$$= \frac{4}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right) - \frac{1}{n^2}\left(\frac{n\,(n+1)}{2}\right)$$
$$= \frac{2}{3n^2}\left(2n^2+3n+1\right) - \frac{1}{2n}(n+1)$$
$$= \frac{5}{6}+\frac{3}{2n}+\frac{2}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{5}{6}+\frac{3}{2n}+\frac{2}{3n^2}\right)$$
$$= \frac{5}{6}$$

(b) $\Delta x = \frac{2}{n}$. We will use right endpoints as evalution points, $x_i = -1 + \frac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[4{x_i}^2 - x_i\right]\frac{2}{n}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[4\left(-1+\frac{2i}{n}\right)^2 - \left(-1+\frac{2i}{n}\right)\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(5 - \frac{18i}{n}+\frac{16i^2}{n^2}\right)$$
$$= \frac{10}{n}\sum_{i=1}^{n}1 - \frac{36}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2$$
$$= 10 - \frac{36}{n^2}\left(\frac{n\,(n+1)}{2}\right) + \frac{32}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right)$$
$$= 10 - \frac{18}{n}(n+1) + \frac{16}{3n^2}\left(2n^2+3n+1\right)$$
$$= \frac{8}{3} - \frac{2}{n} + \frac{16}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{8}{3}-\frac{2}{n}+\frac{16}{3n^2}\right)$$
$$= \frac{8}{3}$$

(c) $\Delta x = \frac{2}{n}$. We will use right endpoints as evaluation points, $x_i = 1 + \frac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[4{x_i}^2 - x_i\right]\frac{2}{n}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[4\left(1+\frac{2i}{n}\right)^2 - \left(1+\frac{2i}{n}\right)\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(3 + \frac{14i}{n}+\frac{16i^2}{n^2}\right)$$
$$= \frac{6}{n}\sum_{i=1}^{n}1 + \frac{28}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2$$
$$= 6 + \frac{28}{n^2}\left(\frac{n\,(n+1)}{2}\right) + \frac{32}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right)$$
$$= 6 + \frac{14}{n}(n+1) + \frac{16}{3n^2}\left(2n^2+3n+1\right)$$
$$= \frac{92}{3} + \frac{30}{n} + \frac{16}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{92}{3}+\frac{30}{n}+\frac{16}{3n^2}\right)$$
$$= \frac{92}{3}$$

15.

n	Left Endpoint	Midpoint	Right Endpoint
10	10.56	10.56	10.56
50	10.662	10.669	10.662
100	10.6656	10.6672	10.6656
500	10.6666	10.6667	10.6666
1000	10.6667	10.6667	10.6667
5000	10.6667	10.6667	10.6667

16.

n	Left Endpoint	Midpoint	Right Endpoint
10	0.91940	1.00103	1.07648
50	0.98421	1.00004	1.01563
100	0.99213	1.00001	1.00783
500	0.99843	1.00000	1.00157
1000	0.99921	1.00000	1.00079
5000	0.99984	1.00000	1.00016

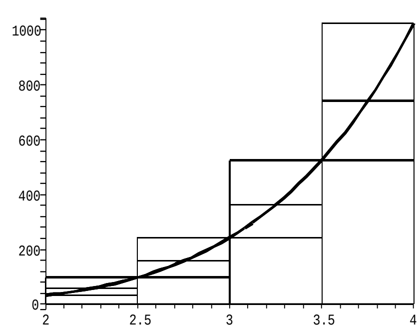
17.

n	Left Endpoint	Midpoint	Right Endpoint
10	15.48000	17.96000	20.68000
50	17.4832	17.9984	18.5232
100	17.7408	17.9996	18.2608
500	17.9480	17.9999	18.0520
1000	17.9740	17.9999	18.0260
5000	17.9948	17.9999	18.0052

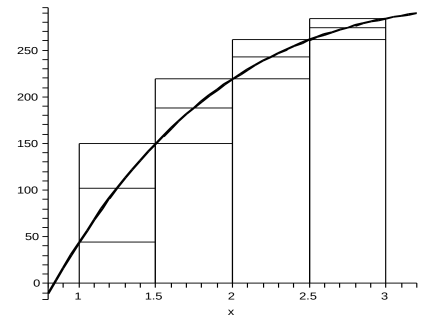
18.

n	Left Endpoint	Midpoint	Right Endpoint
10	-2.20000	-2	-1.80000
50	-2.04000	-2	-1.96000
100	-2.02000	-2	-1.98000
500	-2.00400	-2	-1.99600
1000	-2.00200	-2	-1.99800
5000	-2.00040	-2	-1.99960

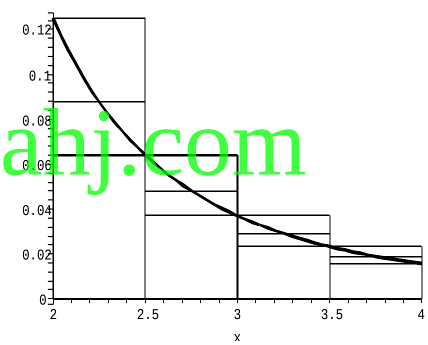
19. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $L < M < A < R$.



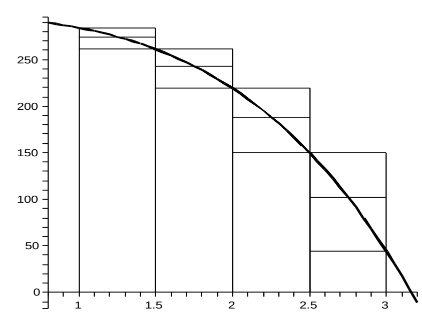
20. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $L < A < M < R$.



21. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $R < A < M < L$.



22. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $R < A < M < L$.



23. There are many possible answers here. One possibility is to use $x = 1/6$ on $[0, 0.5]$ and $x = \sqrt{23}/6$ on $[0.5, 1]$.

24. There are many possible answers here. One possibility is to use $x = 1/4$ on $[0, 0.5]$ and

$x = 25/36$ on $[0.5, 1]$.

25. (a) We subdivide the interval $[a, b]$ into n equal subintervals. If you are located at $a + (b - a)/n$ (the first right endpoint), then each step of distance Δx takes you to a new right endpoint. To arrive at the i-th right endpoint, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + (b-a)/n + (i-1)\Delta x = a + i\Delta x$.

(b) We subdivide the interval $[a, b]$ into n equal subintervals. The first evaluation point is $a + \Delta x/2$. From this evaluation point, each step of distance Δx takes you to a new evaluation point. To arrive at the i-th evaluation point, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + \Delta x/2 + (i-1)\Delta x$
$= a + (i - 1/2)\,\Delta x$, for $i = 1, \ldots, n$.

26. (a) We subdivide the interval $[a, b]$ into n equal subintervals. If you are located at a (the first left endpoint), then each step of distance Δx takes you to a new left endpoint. To arrive at the i-th left endpoint, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + (i-1)\Delta x$.

(b) We subdivide the interval $[a, b]$ into n equal subintervals. The first evaluation point is $a + \Delta x/3$. From this evaluation point, each step of distance Δx takes you to a new evaluation point. To arrive at the i-th evaluation point, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + \Delta x/3 + (i-1)\Delta x$
$= a + (i - 2/3)\,\Delta x$, for $i = 1, \ldots, n$.

27. Consider interval $[2,\, 4]$, then $\Delta x = \dfrac{2}{n}$.
Use right endpoints as evaluation points, $x_i = \left(2 + \dfrac{2i}{n}\right)$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\sqrt{2 + \frac{2i}{n}}\right)\frac{2}{n}\right]$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{2}\left(\sqrt{1 + \frac{i}{n}}\right)\frac{2}{n}\right]$$

Hence,
$$A_2 = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{2}\left(\sqrt{1 + \frac{i}{n}}\right)\frac{2}{n}\right].$$

28. Consider interval$[0,\, 2]$, then $\Delta x = \dfrac{2}{n}$.
Use mid points as evaluation points, $x_i = \dfrac{\left(\frac{2(i-1)}{n} + \frac{2i}{n}\right)}{2}$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\sqrt{\frac{\frac{2(i-1)}{n} + \frac{2i}{n}}{2}}\right)\frac{2}{n}\right]$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\sqrt{\frac{2i - 2 + 2i}{2n}}\right)\frac{2}{n}\right]$$

Hence,
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2i-1}\right)\frac{2}{n}\right].$$

Assume
$i = k + 1$.
$$A = \sum_{k=0}^{n-1} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2\,(k+1) - 1}\right)\frac{2}{n}\right]$$
$$= \sum_{k=1}^{n} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2k+1}\right)\frac{2}{n}\right]$$

hence,
$$A_1 = \sum_{k=1}^{n} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2k+1}\right)\frac{2}{n}\right].$$

29. $U_4 = \dfrac{2}{4}\displaystyle\sum_{i=1}^{4}\left(\frac{i}{2}\right)^2$

$$= \frac{1}{8}\sum_{i=1}^{4} i^2 = \frac{1}{8}\left[1^2 + 2^2 + 3^2 + 4^2\right]$$

$$= \frac{30}{8} = 3.75 \;\; L_4 = \frac{2}{4}\sum_{i=1}^{4}\left(\frac{i-1}{2}\right)^2$$

$$= \frac{1}{8}\sum_{i=1}^{4} i^2 = \frac{1}{8}\left[0^2 + 1^2 + 2^2 + 3^2\right]$$

$$= \frac{14}{8} = 1.75$$

30. The function $f(x) = x^2$ is symmetric on the two intervals $[-2, 0]$ and $[0, 2]$, so the upper sum U_8 is just double the value of U_4 as calculated in Exercise 35, and the same is for L_8. The answers are
$U_8 = 2 \cdot 3.75 = 7.5, L_8 = 2 \cdot 1.75 = 3.5$.

31. (a) $U_n = \dfrac{2}{n}\displaystyle\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2$

$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n} i^2$$

$$= \left(\frac{2}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{4}{3}\frac{n(n+1)(2n+1)}{n^3}$$
$$= \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$
$$\lim_{n\to\infty} U_n = \frac{4}{3}(2) = \frac{8}{3}$$

(b) $L_n = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(i-1)}{n}\right)^2$

$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n}(i-1)^2$$
$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n-1} i^2$$
$$= \left(\frac{2}{n}\right)^3 \frac{(n-1)(n)(2n-1)}{6}$$
$$= \frac{4}{3}\frac{(n-1)(n)(2n-1)}{n^3}$$
$$= \frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$
$$\lim_{n\to\infty} L_n = \frac{4}{3}(2) = \frac{8}{3}$$

32. (a) $U_n = \frac{2}{n}\sum_{i=1}^{n}\left[\left(0+\frac{2}{n}i\right)^3+1\right]$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3+1\right]$$
$$= \left(\frac{2}{n}\right)^4 \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} 1$$
$$= \frac{2^4}{n^4}\left[\frac{n^2(n+1)^2}{4} + \frac{2}{n}(n)\right]$$
$$= \frac{4(n+1)^2}{n^2} + 2$$
$$= \frac{4(n^2+2n+1)}{n^2} + 2$$
$$= 6 + \frac{8}{n} + \frac{4}{n^2}$$
$$\lim_{n\to\infty} U_n = 6$$

(b) $L_n = \frac{2}{n}\sum_{i=0}^{n-1}\left[\left(0+\frac{2}{n}i\right)^3+1\right]$

$$= \frac{2}{n}\sum_{i=0}^{n-1}\left[\left(\frac{2i}{n}\right)^3+1\right]$$
$$= \left(\frac{2}{n}\right)^4 \sum_{i=0}^{n-1} i^3 + \sum_{i=1}^{n} 1$$
$$= \frac{2^4}{n^4}\left[\frac{(n-1)^2n^2}{4} + \frac{2}{n}(n)\right]$$
$$= \frac{4(n-1)^2}{n^2} + 2$$
$$= \frac{4(n^2-2n+1)}{n^2} + 2$$
$$= 6 - \frac{8}{n} + \frac{4}{n^2}$$
$$\lim_{n\to\infty} L_n = 6$$

33. Here, $f(x) = a^2 - x^2$ and interval is $[-a,\, a]$.
Hence $\Delta x = \frac{2a}{n}$.
Use right endpoints as evaluation points, $x_i = \left(-a + \frac{2ai}{n}\right)$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \sum_{i=1}^{n}\left(a^2 - {x_i}^2\right)\Delta x$$
$$= \sum_{i=1}^{n}\left[\left(a^2 - \left(-a+\frac{2ia}{n}\right)^2\right)\frac{2a}{n}\right]$$
$$= \sum_{i=1}^{n}\left[\left(\frac{4ia^2}{n} - \frac{4i^2a^2}{n^2}\right)\frac{2a}{n}\right]$$

$$= \frac{8a^3}{n^2}\sum_{i=1}^{n} i - \frac{8a^3}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{8a^3}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8a^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \frac{4a^3}{n}(n+1) - \frac{4a^3}{3n^3}\left(2n^2+3n+1\right)$$

Now, to compute the exact area, we take the limit as $n\to\infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left[\frac{4a^3}{n}(n+1) - \frac{4a^3}{3n^3}\left(2n^2+3n+1\right)\right]$$
$$= \left(4-\frac{8}{3}\right)a^3 = \frac{4}{3}a^3$$
$$= \frac{2}{3}(2a)\left(a^2\right)$$

34. Here, $f(x) = ax^2$ and interval is $[0,\, b]$.
Hence $\Delta x = \frac{b}{n}$.
Use right endpoints as evaluation points, $x_i = \left(\frac{bi}{n}\right)$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$= \sum_{i=1}^{n} \left(a{x_i}^2\right) \Delta x$
$= \sum_{i=1}^{n} \left[a \left(\frac{bi}{n} \right)^2 \frac{b}{n} \right]$
$= \frac{ab^3}{n^3} \sum_{i=1}^{n} i^2$
$= \frac{ab^3}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right)$
$= \frac{ab^3}{6n^2} \left(2n^2 + 3n + 1\right)$
Now, to compute the exact area, we take the limit as $n \to \infty$:
$A = \lim_{n\to\infty} A_n$
$= \lim_{n\to\infty} \left[\frac{ab^3}{6n^2} \left(2n^2 + 3n + 1\right) \right]$
$= \frac{2ab^3}{6} = \frac{ab^3}{3} = \frac{1}{3} b \left(ab^2\right)$

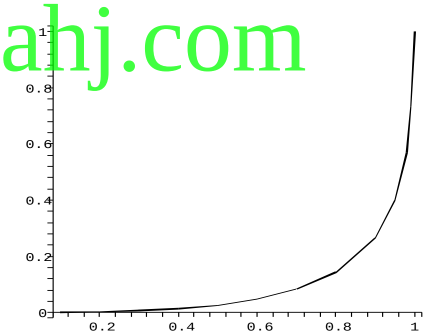
35. Using left hand endpoints:
$L_8 = [f(0.0)+f(0.1)+f(0.2)+f(0.3)+f(0.4)+f(0.5)+f(0.6)+f(0.7)](0.1)$
$= (2.0 + 2.4 + 2.6 + 2.7 + 2.6 + 2.4 + 2.0 + 1.4)(0.1) = 1.81$
Right endpoints:
$R_8 = [f(0.1)+f(0.2)+f(0.3)+f(0.4)+f(0.5)+f(0.6)+f(0.7)+f(0.8)](0.2)$
$= (2.4 + 2.6 + 2.7 + 2.6 + 2.4 + 2.0 + 1.4 + 0.6)(0.1) = 1.67$

36. Using left hand endpoints:
$L_8 = [f().0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1.0)+f(1.2)+f(1.4)](0.2)$
$= (2.0 + 2.2 + 1.6 + 1.4 + 1.6 + 2.0 + 2.2 + 2.4)(0.2) = 3.08$
Right endpoints:
$R_8 = [f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1.0)+f(1.2)+f(1.4)+f(1.6)](0.2)$
$= (2.2 + 1.6 + 1.4 + 1.6 + 2.0 + 2.2 + 2.4 + 2.0)(0.2) = 3.08$

37. Using left hand endpoints:
$L_8 = [f(1.0)+f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)+f(1.6)+f(1.7)](0.1)$
$= (1.8 + 1.4 + 1.1 + 0.7 + 1.2 + 1.4 + 1.82 + 2.4)(0.1) = 1.182$
Right endpoints:
$R_8 = [f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)+f(1.6)+f(1.7)+f(1.8)](0.1)$
$= (1.4 + 1.1 + 0.7 + 1.2 + 1.4 + 1.82 + 2.4 + 2.6)(0.1) = 1.262$

38. Using left hand endpoints:
$L_8 = [f(1.0)+f(1.2)+f(1.4)+f(1.6)+f(1.8)+f(2.0)+f(2.2)+f(2.4)](0.2)$
$= (0.0 + 0.4 + 0.6 + 0.8 + 1.2 + 1.4 + 1.2 + 1.4)(0.2) = 1.40$
Right endpoints:
$R_8 = [f(1.2)+f(1.4)+f(1.6)+f(1.8)+f(2.0)+f(2.2)+f(2.4)+f(2.6)](0.2)$
$= (0.4 + 0.6 + 0.8 + 1.2 + 1.4 + 1.2 + 1.4 + 1.0)(0.2) = 1.60$

39. $A \approx (0.2 - 0.1)(0.002) + (0.3 - 0.2)(0.004) + (0.4 - 0.3)(0.008) + (0.5 - 0.4)(0.014) + (0.6 - 0.5)(0.026) + (0.7 - 0.6)(0.048) + (0.8 - 0.7)(0.085) + (0.9 - 0.8)(0.144) + (0.95 - 0.9)(0.265) + (0.98 - 0.95)(0.398) + (0.99 - 0.98)(0.568) + (1 - 0.99)(0.736) + 1/2 \cdot [(0.1 - 0)(0.002)$
$+ (0.2-0.1)(0.004-0.002) + (0.3-0.2)(0.008-0.004) + (0.4 - 0.3)(0.014 - 0.008) + (0.5 - 0.4)(0.026 - 0.014) + (0.6 - 0.5)(0.048 - 0.026) + (0.7 - 0.6)(0.085 - 0.048) + (0.8 - 0.7)(0.144 - 0.085) + (0.9 - 0.8)(0.265 - 0.144) + (0.95 - 0.9)(0.398 - 0.265) + (0.98 - 0.95)(0.568 - 0.398) + (0.99 - 0.98)(0.736 - 0.568)\ (1 - 0.99)(1 - 0.736)]$
≈ 0.092615 The Lorentz curve looks like:



40. Obviously $G = A_1/A_2$ is greater or equal to 0. From the above figure we see that the Lorentz curve is below the diagonal line $y = x$ on the interval $[0, 1]$, hence the area $A_1 \leq$ the area A_2. Furthermore, $A_2 =$ the area of the triangle formed by the points $(0, 0), (1, 0)$ and $(1, 1)$, hence equal to $1/2$. Now $G = A_1/A_2 = 2A_1$. Using the date in Exercise 33, $G \approx 2 \cdot 0.092615 = 0.185230$.

4.4 The Definite Integral

1. We know that
$\int_0^3 \left(x^3 + x\right) dx \approx \sum_{i=1}^{n} \left(c_i^3 + c_i\right) \Delta x$
Where $c_i = \frac{x_i + x_{i-1}}{2}, x_i = \frac{3i}{n}, n = 6.$

Here $c_i = \dfrac{\frac{3i}{6} + \frac{3(i-1)}{6}}{2} = \dfrac{(2i-1)}{4}$.

$\sum_{i=1}^{n} \left(c_i^3 + c_i\right) . \dfrac{3}{n}$

$= \sum_{i=1}^{6} \left[\dfrac{(2i-1)^3}{64} + \dfrac{(2i-1)}{4} \right] . \dfrac{1}{2}$

$= \left(\dfrac{1}{64} + \dfrac{1}{4} + \dfrac{27}{64} + \dfrac{3}{4} + \dfrac{125}{64} + \dfrac{5}{4} + \dfrac{343}{64} \right.$

$\left. + \dfrac{7}{4} + \dfrac{729}{64} + \dfrac{9}{4} + \dfrac{1331}{64} + \dfrac{11}{4} \right) . \dfrac{1}{2}$

$\Rightarrow \int_0^3 \left(x^3 + x\right) dx \approx 24.47$

2. We know that

$\int_0^3 \sqrt{x^2+1} dx \approx \sum_{i=1}^{n} \sqrt{c_i^2 + 1} \Delta x$

Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = \dfrac{3i}{n}, n = 6.$

Here $c_i = \dfrac{\frac{3i}{6} + \frac{3(i-1)}{6}}{2} = \dfrac{(2i-1)}{4}$.

$\sum_{i=1}^{n} \sqrt{c_i^2 + 1} \left(\dfrac{3}{n} \right)$

$= \sum_{i=1}^{6} \left(\sqrt{\left(\dfrac{2i-1}{4} \right)^2 + 1} \right) \dfrac{1}{2}$

$= \left(\dfrac{\sqrt{17}}{4} + \dfrac{5}{4} + \dfrac{\sqrt{41}}{4} + \dfrac{\sqrt{65}}{4} \right.$

$\left. + \dfrac{\sqrt{97}}{4} + \dfrac{\sqrt{137}}{4} \right) . \dfrac{1}{2}$

$\Rightarrow \int_0^3 \sqrt{x^2+1} dx \approx 5.64$

3. We know that

$\int_0^{\pi} \sin x^2 dx \approx \sum_{i=1}^{n} \left(\sin c_i^2\right) \Delta x.$

Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = \dfrac{i\pi}{n}, n = 6.$

Here$c_i = \dfrac{\frac{\pi i}{6} + \frac{\pi(i-1)}{6}}{2} = \dfrac{(2i-1)\pi}{12}$.

$\sum_{i=1}^{n} \left(\sin c_i^2\right) \left(\dfrac{\pi}{n} \right)$

$= \sum_{i=1}^{6} \left[\sin \left(\dfrac{(2i-1)\pi}{12} \right)^2 \right] . \left(\dfrac{\pi}{6} \right)$

$= \left[\sin \left(\dfrac{\pi}{12} \right)^2 + \sin \left(\dfrac{3\pi}{12} \right)^2 + \sin \left(\dfrac{5\pi}{12} \right)^2 \right.$

$\left. + \sin \left(\dfrac{7\pi}{12} \right)^2 + \sin \left(\dfrac{9\pi}{12} \right)^2 + \sin \left(\dfrac{11\pi}{12} \right)^2 \right] . \dfrac{\pi}{6}$

$\Rightarrow \int_0^{\pi} \sin x^2 dx \approx 0.8685$

4. We know that

$\int_{-2}^{2} e^{-x^2} dx \approx \sum_{i=1}^{n} e^{-c_i^2} \Delta x.$

Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = -2 + \dfrac{4i}{n}, n = 6.$

Here,

$c_i = \dfrac{\left(-2 + \frac{4i}{6}\right) + \left[-2 + \frac{4(i-1)}{6}\right]}{2} = \dfrac{2i-7}{3}.$

$\sum_{i=1}^{n} e^{-c_i^2} \left(\dfrac{4}{n} \right) = \sum_{i=1}^{6} e^{-c_i^2} \left(\dfrac{4}{6} \right)$

$= \left[e^{-25/9} + e^{-1} + e^{-1/9} \right.$

$\left. + e^{-1/9} + e^{-1} + e^{-25/9} \right] . \dfrac{2}{3}$

$= \left[e^{-25/9} + e^{-1} + e^{-1/9} \right] . \dfrac{4}{3}$

$\Rightarrow \int_{-2}^{2} e^{-x^2} dx \approx 1.7665$

5. Notice that the graph of $y = x^2$ is above the x-axis. So,$\int_1^3 x^2 dx$ is the area of the region bounded by $y = x^2$ and the x-axis, between $x = 1$ and $x = 3$.

6. Notice that the graph of $y = e^x$ is above the x-axis. So, $\int_0^1 e^x dx$ is the area of the region bounded by $y = e^x$, and the x-axis, between $x = 0$ and $x = 1$.

7. Notice that the graph of $y = x^2 - 2$ is below the $x-$axis for $|x| \le \sqrt{2}$ above the, $x-$axis for $|x| \ge \sqrt{2}$.

Also,

$\int_0^2 \left(x^2 - 2\right) dx$

$= \int_0^{\sqrt{2}} \left(x^2 - 2\right) dx + \int_{\sqrt{2}}^{2} \left(x^2 - 2\right) dx.$

So, $\int_0^2 \left(x^2 - 2\right) dx$ is the additon of the areas of the regions bounded by $y = x^2 - 2$and the $x-$axis, between $x = 0$ and $x = \sqrt{2}$ (which is below the $x-$axis) and between $x = \sqrt{2}$ and $x = 2$ (which is above the $x-$axis)

8. Notice that the graph of $y = x^3 - 3x^2 + 2x$ is below the x-axis, for $1 \le x \le 2$ and $x \le 0$ and above the x-axis, for all other values of x.

Also,

$\int_0^2 \left(x^3 - 3x^2 + 2x\right) dx$

$= \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx$

$+ \int_1^2 \left(x^3 - 3x^2 + 2x\right) dx$

So, $\int_0^2 \left(x^3 - 3x^2 + 2x\right) dx$ is the additon of the areas of the regions bounded by $y = x^3 - 3x^2 + 2x$ and the x-axis between $x = 0$ and $x = 1$ (which is above the x-axis) and between $x = 1$ and $x = 2$ (which is below the x-axis).

9. For n rectangles, $\Delta x = \dfrac{1}{n}$, $x_i = i\Delta x$.

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \sum_{i=1}^{n} 2x_i\ \Delta x = \frac{1}{n}\sum_{i=1}^{n} 2\left(\frac{i}{n}\right) = \frac{2}{n^2}\sum_{i=1}^{n} i$$
$$= \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{(n+1)}{n}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_0^1 2x dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)}{n} = 1$$

10. For n rectangles, $\Delta x = \dfrac{1}{n}$, $x_i = 1 + i\Delta x$.

$$R_n = \sum_{i=1}^{n} f(x_i)\ \Delta x$$
$$= \sum_{i=1}^{n} 2x_i\ \Delta x = \frac{1}{n}\sum_{i=1}^{n} 2\left(1 + \frac{i}{n}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i$$
$$= \frac{2}{n}(n) + \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)$$
$$= 2 + \frac{(n+1)}{n}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_1^2 2x dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 2 + \frac{(n+1)}{n}$$
$$= 2 + 1 = 3$$

11. For n rectangles,
$\Delta x = \dfrac{2}{n}, x_i = i\Delta x = \dfrac{2i}{n}$.

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \sum_{i=1}^{n} (x_i^2)\Delta x = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2$$
$$= \frac{2}{n}\sum_{i=1}^{n} \frac{4i^2}{n^2} = \frac{8}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \frac{4(n+1)(2n+1)}{3n^2}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_0^2 x^2 dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{3n^2} = \frac{8}{3}$$

12. For n rectangles,
$\Delta x = \dfrac{3}{n}, x_i = i\Delta x = \dfrac{3i}{n}$.

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \sum_{i=1}^{n} (x_i^2 + 1)\Delta x = \frac{3}{n}\sum_{i=1}^{n} 2\left(\frac{3i}{n}\right)^2 + 1$$
$$= \frac{3}{n}\sum_{i=1}^{n} \frac{18i^2}{n^2} + 1$$
$$= \frac{54}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1$$
$$= \frac{54}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{3}{n}\right) n$$
$$= \frac{9(n+1)(2n+1)}{n^2} + 3$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_0^3 (x^2 + 1) dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{n^2} + 3$$
$$= 9 + 3 = 12$$

13. For n rectangles, $\Delta x = \dfrac{2}{n}$,

$$x_i = 1 + i\Delta x = 1 + \frac{2i}{n}$$
$$R_n = \sum_{i=1}^{n} f(x_i)\ \Delta x$$
$$= \sum_{i=1}^{n} (x_i^2 - 3)\Delta x$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(1 + \frac{2i}{n}\right)^2 - 3\right]$$
$$= \sum_{i=1}^{n}\left(\frac{8i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right)$$
$$= \frac{8n(n+1)}{2n^2} + \frac{8n(n+1)(2n+1)}{6n^3} - 4$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_1^3 (x^2 - 3)dx = \lim_{n\to\infty} R_n$$

$$= \frac{8}{2} + \frac{16}{6} - 4 = \frac{8}{3}$$

14. For n rectangles,

$$\Delta x = \frac{4}{n}, x_i = -2 + i\Delta x = -2 + \frac{4i}{n}$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} (x_i^2 - 1)\Delta x$$

$$= \frac{4}{n}\sum_{i=1}^{n}\left(-2 + \frac{4i}{n}\right)^2 - 1$$

$$= \frac{4}{n}\sum_{i=1}^{n}\left(3 - \frac{16i}{n} + \frac{16i^2}{n^2}\right)$$

$$= \frac{12}{n}\sum_{i=1}^{n} 1 - \frac{64}{n^2}\sum_{i=1}^{n} i + \frac{64}{n^3}\sum_{i=1}^{n} i^2$$

$$= \left(\frac{12}{n}\right)n - \frac{64}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$+ \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 12 - \frac{32(n+1)}{n} + \frac{32(n+1)(2n+1)}{3n^2}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_{-2}^{2} (x^2 - 1)dx = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty}\left[12 - \frac{32(n+1)}{n} + \frac{32(n+1)(2n+1)}{3n^2}\right]$$

$$= 12 - 32 + \frac{64}{3} = \frac{4}{3}$$

15. Notice that the graph of $y = 4 - x^2$ is above the x-axis between $x = -2$ and $x = 2$:

$$\int_{-2}^{2} (4 - x^2)dx$$

16. Notice that the graph of $y = 4x - x^2$ is above the x-axis between $x = 0$ and $x = 4$:

$$\int_0^4 (4x - x^2)dx$$

17. Notice that the graph of $y = x^2 - 4$ is below the x-axis between $x = -2$ and $x = 2$. Since we are asked for area and the area in question is below the x-axis, we have to be a bit careful.

$$\int_{-2}^{2} -(x^2 - 4)dx$$

18. Notice that the graph of $y = x^2 - 4x$ is below the x-axis between $x = 0$ and $x = 4$. Since we are asked for area and the area in question is below the x-axis, we have to be a bit careful.

$$\int_0^4 -(x^2 - 4x)dx$$

19. $\displaystyle\int_0^\pi \sin x dx$

20. $\displaystyle -\int_{-\pi/2}^{0} \sin x dx + \int_0^{\pi/4} \sin x dx$

21. The total distance is the total area under the curve whereas the total displacement is the signed area under the curve. In this case, from $t = 0$ to $t = 4$, the function is always positive so the total distance is equal to the total displacement. This means we want to compute the definite integral $\displaystyle\int_0^4 40(1 - e^{-2t})dt$. We compute various right hand sums for different values of n:

n	R_n
10	146.9489200
20	143.7394984
50	141.5635684
100	140.7957790
500	140.1662293
1000	140.0865751

It looks like these are converging to about 140. So, the total distance traveled is approximately 140 and the final position is $s(b) \approx s(0) + 140 = 0 + 140 = 140$.

22. The total distance is the total area under the curve whereas the total displacement is the signed area under the curve. In this case, from $t = 0$ to $t = 4$, the function is always positive so the total distance is equal to the total displacement. This means we want to compute the definite integral $\int_0^4 30e^{-t/4}\, dt$. We compute various right hand sums for different values of n:

n	R_n
10	72.12494524
20	73.97390774
50	75.09845086
100	75.47582684
500	75.77863788
1000	75.81654616

It looks like these are converging to about 75.8. So, the total distance traveled is approximately 75.8 and the final position is $s(b) \approx s(0) + 75.8 = -1 + 75.8 = 74.8$.

23. $\int_0^4 f(x)dx$
$= \int_0^1 f(x)dx + \int_1^4 f(x)dx$
$= \int_0^1 2xdx + \int_1^4 4dx$
$\int_0^1 2xdx$ is the area of a triangle with base 1 and height 2 and therefore has $area = \frac{1}{2}(1)(2) = 1$.
$\int_1^4 4dx$ is the area of a rectangle with base 3 and height 4 and therefore has $area = (3)(4) = 12$.
Therefore
$\int_0^4 f(x)dx = 1 + 12 = 13$

24. $\int_0^4 f(x)dx$
$= \int_0^2 f(x)dx + \int_2^4 f(x)dx$
$= \int_0^2 2dx + \int_2^4 3xdx$
$\int_0^2 2dx$ is the area of a square with base 2 and height 2 (it is, after all, a square) and therefore has $area = 4$.
$\int_2^4 3xdx$ is a trapezoid with height 3 and bases 6 and 12 and therefore has area (using the formula in the front of the text)
$area = \frac{1}{2}(6 + 12)(2) = 18$.
Therefore
$\int_0^4 f(x)dx = 4 + 18 = 22$

25. $f_{ave} = \frac{1}{4}\int_0^4 (2x+1)dx$
$= \frac{1}{4}\lim_{n\to\infty} \sum_{i=1}^n \frac{4}{n}\left(\frac{8i}{n} + 1\right)$
$= \lim_{n\to\infty} \left(\frac{8n(n+1)}{2n^2} + 1\right)$
$= 4 + 1 = 5$

26. $f_{ave} = \frac{1}{1}\int_0^1 (x^2 + 2x)dx$
$= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}\left(\frac{i^2}{n^2} + \frac{2i}{n}\right)$
$= \lim_{n\to\infty} \left(\frac{n(n+1)(2n+1)}{6n^3} + \frac{2n(n+1)}{n^2}\right)$
$= \frac{2}{6} + 2 = \frac{7}{3}$

27. $f_{ave} = \frac{1}{3-1}\int_1^3 (x^2 - 1)dx$
$= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}\left[\left(1 + \frac{2i}{n}\right)^2 - 1\right]$
$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{4i}{n} + \frac{4i^2}{n^2}\right)$
$= \lim_{n\to\infty} \left(\frac{4n(n+1)}{2n^2} + \frac{4n(n+1)(2n+1)}{6n^3}\right)$
$= 2 + \frac{4}{3} = \frac{10}{3}$

28. $f_{ave} = \frac{1}{1-0}\int_0^1 (2x - 2x^2)dx$
$= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}\left[2\left(\frac{i}{n}\right) - 2\left(\frac{i}{n}\right)^2\right]$
$= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}\left(\frac{2i}{n} + \frac{2i^2}{n^2}\right)$
$= \lim_{n\to\infty} \left(\frac{2n(n+1)}{2n^2} + \frac{2n(n+1)(2n+1)}{6n^3}\right)$
$= 1 + \frac{2}{3} = \frac{5}{3}$

29. The function $f(x) = 3\cos x^2$ is decreasing on $[\pi/3, \pi/2]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(\pi/3) = 3\cos(\pi^2/9)$. The minimum occurs at the right endpoint and is $f(\pi/2) = 3\cos(\pi^2/4)$. Using these to estimate the value of the integral gives the following inequality:

$$\frac{\pi}{6} \cdot (3\cos\frac{\pi^2}{4}) \le \int_{\pi/3}^{\pi/2} 3\cos x^2 dx \le \frac{\pi}{6} \cdot (3\cos\frac{\pi^2}{9})$$

$$-1.23 \le \int_{\pi/3}^{\pi/2} 3\cos x^2 dx \le 0.72$$

30. The function $f(x) = e^{-x^2}$ is decreasing on $[0, 1/2]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(0) = 1$. The minimum occurs at the right endpoint and is $f(1/2) = e^{-1/4}$. Using these to estimate

the value of the integral gives the following inequality:
$\frac{1}{2}(e^{-1/4}) \le \int_0^{1/2} e^{-x^2} dx \le \frac{1}{2}(1)$
$0.3894 \le \int_0^{1/2} e^{-x^2} dx \le 0.5$

31. The function $f(x) = \sqrt{2x^2+1}$ is increasing on $[0,2]$. Therefore, on this interval, the maximum occurs at the right endpoint and is $f(2) = 3$. The minimum occurs at the left endpoint and is $f(0) = 1$. Using these to estimate the value of the integral gives the following inequality:
$(2)(1) \le \int_0^2 \sqrt{2x^2+1}dx \le (2)(3)$
$2 \le \int_0^2 \sqrt{2x^2+1}dx \le 6$

32. The function $f(x) = \frac{3}{x^3+2}$ is decreasing on $[-1,1]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(-1) = 3$. The minimum occurs at the right endpoint and is $f(1) = 1$. Using these to estimate the value of the integral gives the following inequality:
$(2)(1) \le \int_{-1}^1 \frac{3}{x^3+2}dx \le (2)(3)$
$2 \le \int_{-1}^1 \frac{3}{x^3+2}dx \le 6$

33. We are looking for a value c, such that
$f(c) = \frac{1}{2-0}\int_0^2 3x^2 dx$
Since $\int_0^2 3x^2 dx = 8$, we want to find c so that $f(c) = 4$ or, $3c^2 = 4$
Solving this equation using the quadratic formula gives $c = \pm\frac{2}{\sqrt{3}}$
We are interested in the value that is in the interval $[0,2]$, so $c = \frac{2}{\sqrt{3}}$.

34. We are looking for a value c, such that
$f(c) = \frac{1}{1-(-1)}\int_{-1}^1 (x^2-2x)dx$
Since $\int_{-1}^1 (x^2-2x)dx = \frac{2}{3}$, we want to find c so that $f(c) = \frac{1}{3}$ or, $c^2 - 2c = \frac{1}{3}$

Solving this equation using the quadratic formula gives $c = \frac{3 \pm 2\sqrt{3}}{3}$

We are interested in the value that is in the interval $[-1,1]$, so $c = \frac{3-2\sqrt{3}}{3}$.

35. (a) $\int_0^2 f(x)dx + \int_2^3 f(x)dx = \int_0^3 f(x)dx$

(b) $\int_0^3 f(x)dx - \int_2^3 f(x)dx = \int_0^2 f(x)dx$

36. (a) $\int_0^2 f(x)dx + \int_2^1 f(x)dx = \int_0^1 f(x)dx$

(b) $\int_{-1}^2 f(x)dx + \int_2^3 f(x)dx = \int_{-1}^3 f(x)dx$

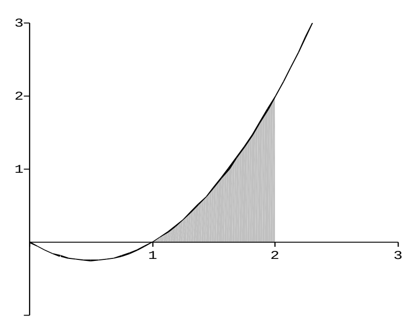
37. (a) $\int_1^3 (f(x) + g(x))\, dx$
$= \int_1^3 f(x)\, dx + \int_1^3 g(x)\, dx$
$= 3 + (-2) = 1$

(b) $\int_1^3 (2f(x) - g(x))\, dx$
$= 2\int_1^3 f(x)\, dx - \int_1^3 g(x)\, dx$
$= 2(3) - (-2) = 8$

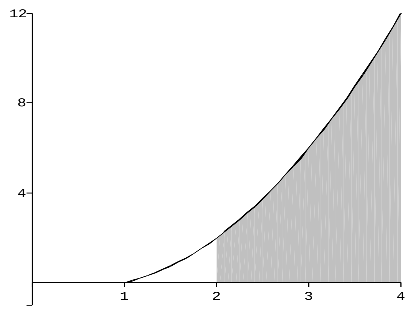
38. (a) $\int_1^3 (f(x) - g(x))\, dx$
$= \int_1^3 f(x)\, dx - \int_1^3 g(x)\, dx$
$= 3 - (-2) = 5$

(b) $\int_1^3 (4g(x) - 3f(x))\, dx$
$= 4\int_1^3 g(x)\, dx - 3\int_1^3 f(x)\, dx$
$= 4(-2) - 3(3) = -17$

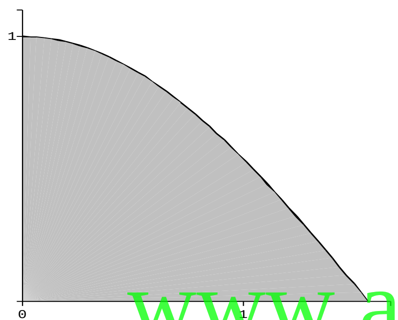
39. (a)



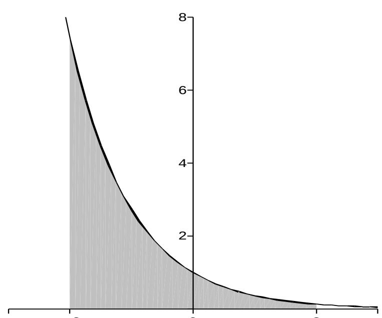
(b)



40. (a)



(b)



41. (a) Notice that $x^2 \sin x$ is a continuous function for all values of x and
for $1 \le x \le 2$,
$\sin x \le x^2 \sin x \le 4 \sin x$.
On using theorem 4.3,we get

$$\int_1^2 \sin x dx \le \int_1^2 x^2 \sin x dx \le \int_1^2 4 \sin x dx$$

$$(\cos 1 - \cos 2) \le \int_1^2 x^2 \sin x dx \le 4 (\cos 1 - \cos 2)$$

(b) Notice that $x^2 \sin x$ is a continuous function for all values of x and
for $1 \le x \le 2, x^2 \sin 1 \le x^2 \sin x \le x^2$.
On using theorem 4.3,we get

$$\sin 1 \int_1^2 x^2 dx \le \int_1^2 x^2 \sin x dx \le \int_1^2 x^2 dx$$

$$\sin 1 \left. \frac{x^3}{3} \right|_1^2 \le \int_1^2 x^2 \sin x dx \le \left. \frac{x^3}{3} \right|_1^2$$

$$\frac{7}{3} \sin 1 \le \int_1^2 x^2 \sin x dx \le \frac{7}{3}$$

(c) Let us evaluate $\int_1^2 x^2 \sin x dx$

using $\int_1^2 x^2 \sin x dx \approx \sum_{i=1}^n c_i^2 \sin c_i \Delta x$

and $n = 6$

Where $c_i = \frac{x_i + x_{i-1}}{2}, x_i = 1 + \frac{i}{6}$,

Here $c_i = \frac{2 + \frac{i}{6} + \frac{(i-1)}{6}}{2}$

$= \frac{(2i + 11)}{12}$

$$\sum_{i=1}^n (c_i^2 \sin c_i) \left(\frac{1}{n} \right)$$

$$= \left[\left(\frac{13}{12} \right)^2 \sin \left(\frac{13}{12} \right) + \left(\frac{15}{12} \right)^2 \sin \left(\frac{15}{12} \right) + \left(\frac{17}{12} \right)^2 \sin \left(\frac{17}{12} \right) + \left(\frac{19}{12} \right)^2 \sin \left(\frac{19}{12} \right) + \left(\frac{21}{12} \right)^2 \sin \left(\frac{21}{12} \right) + \left(\frac{23}{12} \right)^2 \sin \left(\frac{23}{12} \right) \right] \cdot \frac{1}{6}$$

Therefore, $\int_1^2 x^2 \sin x dx \approx 2.2465$

$$(\cos 1 - \cos 2) \le \int_1^2 x^2 \sin x dx \le 4 (\cos 1 - \cos 2)$$

$\Rightarrow 0.9564 \le 2.2465 \le 3.8257$

and

$$\frac{7}{3} \sin 1 \le \int_1^2 x^2 \sin x dx \le \frac{7}{3}$$

$\Rightarrow 1.9634 \le 2.2465 \le 2.3333$

The second inequality gives a range which is more closer to the value of the integral. Therefore, part **(b)** is more useful than part **(a).**

42. Notice that $x^2 e^{-\sqrt{x}}$ is a continuous function for all values of $x \ge 0$.
For $1 \le x \le 2$,

$e^{-\sqrt{2}} \le e^{-\sqrt{x}} \le e^{-1}$

Therefore $x^2e^{-\sqrt{2}} \le x^2e^{-\sqrt{x}} \le x^2e^{-1}$

Thus, on using theorem 4.3.

$$\int_1^2 x^2e^{-\sqrt{2}}dx \le \int_1^2 x^2e^{-\sqrt{x}}dx \le \int_1^2 x^2e^{-1}dx$$

$$e^{-\sqrt{2}}\frac{x^3}{3}\bigg|_1^2 \le \int_1^2 x^2e^{-\sqrt{x}}dx \le e^{-1}\frac{x^3}{3}\bigg|_1^2$$

$$\frac{7}{3}e^{-\sqrt{2}} \le \int_1^2 x^2e^{-\sqrt{x}}dx \le \frac{7}{3}e^{-1}$$

$$0.5672 \le \int_1^2 x^2e^{-\sqrt{x}}dx \le 0.8583$$

43. This is just a restatement of the Integral Mean Value Theorem.

44. Let $c = \frac{a+b}{2}$. By definition,

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x.$$

We can choose n to be always even, so that $n = 2m$, and

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{m\to\infty}\sum_{i=1}^{m} f(c_i)\Delta x + \lim_{m\to\infty}\sum_{i=m+1}^{n} f(c_i)\Delta x$$

$$= \int_a^c f(x)dx + \int_c^b f(x)dx$$

45. Between $x = 0$ and $x = 2$, the area below the x-axis is much less than the area above the x-axis. Therefore $\int_0^2 f(x)dx > 0$

46. Between $x = 0$ and $x = 2$, the area above the x-axis is much greater than the area below the x-axis. Therefore $\int_0^2 f(x)dx > 0$

47. Between $x = 0$ and $x = 2$, the area below the x-axis is slightly greater than the area above the x-axis. Therefore $\int_0^2 f(x)dx < 0$

48. Between $x = 0$ and $x = 2$, the area below the x-axis is much greater than the area above the x-axis. Therefore$\int_0^2 f(x)dx < 0$

49. $\int_0^2 3xdx = \frac{1}{2}bh = \frac{1}{2}(2)(6) = 6$

50. $\int_1^4 2xdx = \frac{1}{2}(a+b)h = \frac{1}{2}(2+8)(3)$

$= 15$

51. $\int_0^2 \sqrt{4-x^2} = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi\left(2^2\right) = \pi$

52. $\int_{-3}^0 \sqrt{9-x^2}dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi 3^2$

$= \frac{9\pi}{4}$

53. (a) Given limit

$$\lim_{n\to\infty}\frac{1}{n}\left[\sin\left(\frac{\pi}{n}\right) + ... + \sin\left(\frac{n\pi}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}\sin\left(\frac{i\pi}{n}\right)\right]$$

We know that

$$\lim_{x\to\infty}\left[\sum_{i=1}^{n} f(c_i)\Delta x\right] = \int_a^b f(x)dx$$

Where $c_i = a + i\Delta x$ and $\Delta x = \left(\frac{b-a}{n}\right)$

On comparision,we get

$c_i = \frac{i}{n}, \Delta x = \frac{1}{n}$ and

$f(x) = \sin(\pi x) \Rightarrow a = 0, b = 1$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}\sin\left(\frac{i\pi}{n}\right)\right] = \int_0^1 \sin(\pi x)dx$$

(b) Given limit

$$= \lim_{n\to\infty}\left[\frac{n+1}{n^2} + \frac{n+2}{n^2} + ... + \frac{2n}{n^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}\frac{n+i}{n}\right]$$

We know that

$$\lim_{x\to\infty}\left[\sum_{i=1}^{n} f(c_i)\Delta x\right] = \int_a^b f(x)dx$$

Where $c_i = a + i\Delta x$ and $\Delta x = \left(\frac{b-a}{n}\right)$

On comparision,we get

$c_i = \frac{i}{n}, \Delta x = \frac{1}{n}$ and $f(x) = 1 + x$

$\Rightarrow a = 0, b = 1$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}\frac{n+i}{n}\Delta x\right] = \int_0^1 (1+x)dx$$

(c) Given limit

$$\lim_{n\to\infty}\left[\frac{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + ... + f\left(\frac{n}{n}\right)}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\right]$$

We know that

$$\lim_{x\to\infty}\left[\sum_{i=1}^{n} f(c_i)\Delta x\right] = \int_a^b f(x)dx$$

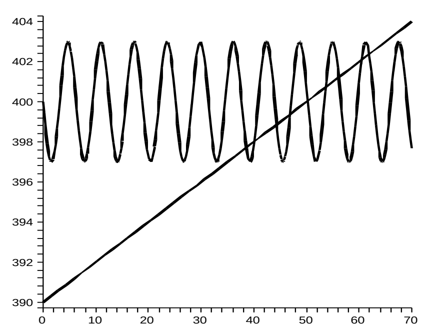
Where $c_i = a + i\Delta x$ and $\Delta x = \left(\frac{b-a}{n}\right)$

On comparision,we get
$c_i = \dfrac{i}{n}$ and $\Delta x = \frac{1}{n}$
$\Rightarrow a = 0, b = 1$
Therefore,
$$\lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\right] = \int_0^1 f(x)dx$$

54. $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = v$
$\displaystyle\int_a^b f(x)dx = v(b-a)$
and
$\dfrac{1}{c-b}\displaystyle\int_b^c f(x)dx = w$
$\displaystyle\int_b^c f(x)dx = w(c-b)$
The average value of f over $[a,c]$ is
$\dfrac{1}{c-a}\displaystyle\int_a^c f(x)dx$
$= \dfrac{1}{c-a}\left[\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx\right]$
$= \dfrac{1}{c-a}[v(b-a) + w(c-b)]$
$= \dfrac{v(b-a) + w(c-b)}{c-a}$

55. Since $b(t)$ represents the birthrate (in births per month), the total number of births from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} b(t)\, dt$.
Similarly, the total number of deaths from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} a(t)\, dt$.
Of course, the net change in population is the number of birth minus the number of deaths:
Population Change
$=$ Births $-$ Deaths
$= \displaystyle\int_0^{12} b(t)\, dt - \int_0^{12} a(t)\, dt$
$= \displaystyle\int_0^{12} [b(t) - a(t)]\, dt.$
Next we solve the inequality
$410 - 0.3t > 390 + 0.2t$
$20 > 0.5t$ then $t < 40$ months .
Therefore $b(t) > a(t)$ when $t < 40$ months. The population is increasing when the birth rate is greater than the death rate, which is during the first 40 month. After 40 months, the population is decreasing. The population-would reach a maximum at $t = 40$ months.

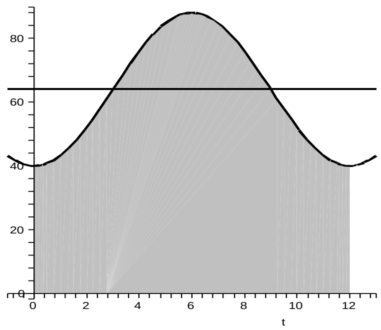
56. Since $b(t)$ represents the birthrate (in births per month), the total number of births from time $t = 0$ to $t = 12$ is given by the integral $\displaystyle\int_0^{12} b(t)dt.$
Similarly, the total number of deaths from time $t = 0$ to $t = 12$ is given by the integral $\displaystyle\int_0^{12} a(t)dt.$
Of course, the net change in population is the number of birth minus the number of deaths:
Population Change
$=$ Births $-$ Deaths
$= \displaystyle\int_0^{12} b(t)dt - \int_0^{12} a(t)dt$
$= \displaystyle\int_0^{12} [b(t) - a(t)]dt.$
By graphing $b(t)$ and $a(t)$ we see that their graphs intersect 9 times, at
$t \approx 38.5, 40.1, 44.4, 46.9, 50.2, 53.6,$
$56.1, 60.5, 61.9.$
This tells us that we have $b(t) > a(t)$ on the intervals
$(0, 38.5), (40.1, 44.4), (46.9, 50.2),$
$(53.6, 56.1), (60.5, 61.9).$
The maximum population will occur when $t = 50.2$.



57. From $PV = 10$ we get $P(V) = 10/V$. By definition,
$\displaystyle\int_2^4 P(V)\, dV = \int_2^4 \frac{10}{V} dV$
$= \displaystyle\sum_{i=1}^{n} \frac{2}{n} \cdot \frac{10}{2 + \frac{2i}{n}}$
An estimate of the value of this integral is setting $n = 100$, and then the integral ≈ 6.93

58. The average temperature over the year is
$\dfrac{1}{12}\displaystyle\int_0^{12} 64 - 24\cos\left(\frac{\pi}{6}t\right)\, dt$. If you look at the graphs $T(t)$ and $f(t) = 64$ you should be able to see that the area under $T(t)$ and $f(t)$ between $t = 0$ to $t = 12$ are equal. This means

that the average temperature is 64.



59. Since r is the rate at which items are shipped, rt is the number of items shipped between time 0 and time t. Therefore, $Q - rt$ is the number of items remaining in inventory at time t. Since $Q - rt = 0$ when $t = Q/r$, the formula is valid for $0 \le t \le Q/r$. The average value of $f(t) = Q - rt$ on the time interval $[0, Q/r]$ is

$$\frac{1}{Q/r - 0}\int_0^{Q/r} f(t)dt$$
$$= \frac{r}{Q}\int_0^{Q/r}(Q - rt)dt$$
$$= \frac{r}{Q}\left[Qt - \frac{1}{2}rt^2\right]_0^{Q/r}$$
$$= \frac{r}{Q}\left[\frac{Q^2}{r} - \frac{r}{2}\frac{Q^2}{r^2}\right]$$
$$= \frac{r}{Q}\left[\frac{Q^2}{2r}\right] = \frac{Q}{2}.$$

60. $f(Q) = c_0\dfrac{D}{Q} + c_c\dfrac{Q}{2}$

$f'(Q) = -\dfrac{c_0 D}{Q^2} + \dfrac{c_c}{2}$

Setting $f'(Q) = 0$ gives

$\dfrac{c_0 D}{Q^2} = \dfrac{c_c}{2}$

$Q = \sqrt{\dfrac{2c_0 D}{c_c}}$. This is the right answer of Q minimizing the total cost $f(Q)$, since when the value of Q is very small, the value of D/Q will get very big, and when the value of Q is very small, the value of $Q/2$ will get very big. This means that the function $f(Q)$ is decreasing on the interval $[0, \sqrt{2c_0 D/c_c}]$ and increasing on the interval $[\sqrt{2c_0 D/c_c}, \infty]$. When $Q = \sqrt{2c_0 D/c_c}$,

$$c_0\frac{D}{Q} = \frac{c_0 D}{\sqrt{\frac{2c_0 D}{c_c}}} = c_c\frac{\sqrt{\frac{2c_0 D}{c_c}}}{2} = c_c\frac{Q}{2}.$$

61. Delivery is completed in time Q/p, and since in that time Qr/p items are shipped, the inventory when delivery is completed is

$Q - \dfrac{Qr}{p} = Q\left(1 - \dfrac{r}{p}\right)$.

The inventory at any time is given by

$$g(t) = \begin{cases} (p-r)t & \text{for} \quad t \in \left[0, \frac{Q}{p}\right] \\ Q - rt & \text{for} \quad t \in \left[\frac{Q}{p}, \frac{Q}{r}\right] \end{cases}$$

The graph of g has two linear pieces. The average value of g over the interval $[0, Q/r]$ is the area under the graph (which is the area of a triangle of base Q/r and height $Q(1-r/p)$) divided by the length of the interval (which is the base of the triangle). Thus the average value of the function is $(1/2)bh$ divided by b, which is

$(1/2)h = (1/2)Q(1 - r/p)$.

This time the total cost is

$f(Q) = c_0\dfrac{D}{Q} + c_c\dfrac{Q}{2}(1 - \dfrac{r}{p})$

$f'(Q) = -\dfrac{c_0 D}{Q^2} + \dfrac{c_c(1 - \frac{r}{p})}{2}$

$f'(Q) = 0$ gives $\dfrac{c_0 D}{Q^2} = \dfrac{c_c}{2}(1 - \dfrac{r}{p})$

$Q = \sqrt{\dfrac{2c_0 D}{c_c(1 - r/p)}}$.

The order size to minimize the total cost is

$Q = \sqrt{\dfrac{2c_0 D}{c_c(1 - r/p)}}$.

62. Use the result from Exercise 60,

$Q = \sqrt{\dfrac{2c_0 D}{c_c}}$

$= \sqrt{\dfrac{2(50,000)(4000)}{3800}} \approx 324.44$.

Since this quantity already takes advantage of largest possible discount, the order size that minimizes the total cost is about 324.44 items.

63. The maximum of

$F(t) = 9 - 10^8(t - 0.0003)^2$

occurs when $10^8(t - 0.0003)^2$ reaches its minimum, that is, when $t = 0.0003$. At that time

$F(0.0003) = 9$ thousand pounds.

We estimate the value of

$\displaystyle\int_0^{0.0006}[9 - 10^8(t - 0.0003)^2]\,dt$ using midpoint sum and $n = 20$, and get $m\Delta v \approx 0.00360$ thousand pound-seconds, so $\Delta v \approx 360$ ft per second.

64. The impulse-momentum equation of Prob-

lem 65 gives $5\Delta v$
$= \int_0^{0.4} (1000 - 25,000(t-0.2)^2)\,dt$
$= \int_0^{0.4} (-25000t^2 + 10000t)\,dt$
Using a midpoint sum and $n = 20$ gives an approximation for this integral of 267.0. This means $5\Delta v \approx 267$ and $\Delta v \approx 53.4$ m/s

4.5 The Fundamental Theorem of Calculus

1. $\int_0^2 (2x-3)dx = (x^2 - 3x)\Big|_0^2 = -2$

2. $\int_0^3 (x^2-2)dx = \left(\frac{x^3}{3} - 2x\right)\Big|_0^3 = 3$

3. $\int_{-1}^1 (x^3+2x)dx = \left(\frac{x^4}{4} + x^2\right)\Big|_{-1}^1 = 0$

4. $\int_0^2 (x^3+3x-1)\,dx$
$= \left(\frac{x^4}{4} - \frac{3x^2}{2} - x\right)\Big|_0^2 = -4$

5. $\int_1^4 \left(x\sqrt{x} + \frac{3}{x}\right)dx$
$= \left(\frac{2}{5}x^{5/2} + 3\log x\right)\Big|_1^4$
$= \frac{2}{5}\cdot 32 + 3\log 4 - \frac{2}{5}.1 - 3\log 1$
$= \frac{62}{5} + 3\log 4$

6. $\int_1^2 \left(4x - \frac{2}{x^2}\right)dx = \left(2x^2 + \frac{2}{x}\right)\Big|_1^2 = 5$

7. $\int_0^1 (6e^{-3x}+4)\,dx = \left(\frac{6e^{-3x}}{-3} + 4x\right)\Big|_0^1$
$= -\frac{2}{e^3} + 4 + 2 - 0 = -\frac{2}{e^3} + 6$

8. $\int_0^2 \left(\frac{e^{2x} - 2e^{3x}}{e^{3x}}\right)dx$
$= \int_0^2 (e^{-x} - 2)\,dx = (-e^{-x} - 2x)\Big|_0^2$
$= -\frac{1}{e^2} - 3$

9. $\int_{\pi/2}^{\pi} (2\sin x - \cos x)dx = -2\cos x - \sin x\Big|_{\pi/2}^{\pi}$
$= 3$

10. $\int_{\pi/4}^{\pi/2} 3\csc x\cot x dx = (-3\csc x)\Big|_{\pi/4}^{\pi/2}$
$= -3 + 3\sqrt{2}$

11. $\int_0^{\pi/4} (\sec t\tan t)\,dt = \sec t\Big|_0^{\pi/4}$
$= \sqrt{2} - 1$

12. $\int_0^{\pi/4} \sec^2 t dt = \tan t\Big|_0^{\pi/4} = 1$

13. $\int_0^{1/2} \frac{3}{\sqrt{1-x^2}}dx = 3\sin^{-1}x\Big|_0^{1/2}$
$= 3\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{2}$

14. $\int_{-1}^1 \frac{4}{1+x^2}dx = 4\arctan x\Big|_{-1}^1 = 2\pi$

15. $\int_1^4 \frac{t-3}{t}dt$
$= \int_1^4 (1 - 3t^{-1})\,dt = (t - 3\ln|t|)\Big|_1^4$
$= 3 - 3\ln 4$

16. $\int_0^4 t\,(t-2)\,dt = \left(\frac{t^3}{3} - t^2\right)\Big|_0^4 = \frac{16}{3}$

17. $\int_0^t \left(e^{x/2}\right)^2 dx = (e^x)\Big|_0^t = e^t - 1$

18. $\int_0^t (\sin^2 x + \cos^2 x)\,dx$
$= \int_0^t 1dx = (x)\Big|_0^t = t$

19. The graph of $y = 4 - x^2$ is above the x-axis over the interval $[-2, 2]$.
$\int_{-2}^2 (4-x^2)dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^2 = \frac{32}{3}$

20. The graph of $y = x^2 - 4x$ is below the x-axis over the interval $[0, 4]$.
$\int_0^4 -(x^2-4x)dx = \left(-\frac{x^3}{3} + 2x^2\right)\Big|_0^4 = \frac{32}{3}$

21. The graph of $y = x^2$ is above the x-axis over the interval $[0, 2]$.
$\int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$

22. The graph of $y = x^3$ is above the x-axis over the interval
$[0, 3]$.
$\int_0^3 x^3 dx = \left(\frac{x^4}{4}\right)\Big|_0^3 = \frac{81}{4}$

23. The graph of $y = \sin x$ is above the x-axis over the interval $[0, \pi]$.
$\int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = 2$

24. The graph of $y = \sin x$ is below the x-axis over the interval $\left[-\frac{\pi}{2}, 0\right]$ and above the x-axis over the interval $\left[0, \frac{\pi}{4}\right]$. Hence we need to compute two seperate integrals and add them together:

$$\int_{-\pi/2}^0 -\sin x dx + \int_0^{\pi/4} \sin x dx$$
$$= 1 + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \frac{1}{\sqrt{2}}.$$

25. $f'(x) = x^2 - 3x + 2$

26. $f'(x) = x^2 - 3x - 4$

27. $f'(x) = \left(e^{-\left(x^2\right)^2} + 1\right)\frac{d}{dx}\left(x^2\right)$
$= 2x\left(e^{-x^4} + 1\right)$

28. $f'(x) = -\sec x$

29. $f(x) = \int_{e^x}^0 \sin t^2 dt + \int_0^{2-x} \sin t^2 dt$
$f'(x) = -\sin e^{2x}\frac{d}{dx}(e^x)$
$+ \sin(2-x)^2 \frac{d}{dx}(2-x)$
$= -e^x \sin e^{2x} - \sin(2-x)^2$

30. $f(x) = \int_{2-x}^0 e^{2t} dt + \int_0^{xe^x} e^{2t} dt$
$f'(x) = -e^{2(2-x)}\frac{d}{dx}(2-x)$
$+ e^{2(xe^x)}\frac{d}{dx}(xe^x)$
$= e^{4-2x} + e^{2xe^x}(xe^x + e^x)$

31. $f(x) = \int_{x^2}^0 \sin(2t)\, dt + \int_0^{x^3} \sin(2t)\, dt$
$f'(x) = -\sin\left(2x^2\right)\frac{d}{dx}\left(x^2\right)$
$+ \sin\left(2x^3\right)\frac{d}{dx}\left(x^3\right)$
$= -2x\sin\left(2x^2\right) + 3x^2\sin\left(2x^3\right)$

32. $f(x) == \int_{3x}^0 \left(t^2+4\right) dt + \int_0^{\sin x} \left(t^2+4\right) dt$
$= -\int_0^{3x} \left(t^2+4\right) dt + \int_0^{\sin x} \left(t^2+4\right) dt$
$f'(x) = -\left(9x^2+4\right)\frac{d}{dx}(3x)$
$+ \left(\sin^2 x + 4\right)\frac{d}{dx}(\sin x)$
$= -27x^2 - 12 + \sin^2 x \cos x + 4\cos x$

33. $s(t) = 40t + \cos t + c$,
$s(0) = 0 + \cos 0 + c = 2$
so therefore $c = 1$ and $s(t) = 40t + \cos t + 1$.

34. $s(t) = 10e^t + c$,
$s(0) = 10 + c = 2$
so therefore $c = -8$ and $s(t) = 10e^{-t} - 8$.

35. $v(t) = 4t - \frac{t^2}{2} + c_1$,
$v(0) = c_1 = 8$
so therefore $c_1 = 8$ and $v(t) = 4t - \frac{t^2}{2} + 8$.
$s(t) = 2t^2 - \frac{t^3}{6} + 8t + c_2$,
$s(0) = c_2 = 0$
so therefore $c_2 = 0$ and $s(t) = 2t^2 - \frac{t^3}{6} + 8t$.

36. $v(t) = 16t - \frac{t^3}{3} + c_1$,
$v(0) = c_1 = 0$
so therefore $c_1 = 0$ and
$v(t) = 16t - \frac{t^3}{3}$.
$s(t) = 8t^2 - \frac{t^4}{12} + c_2$,
$s(0) = c_2 = 30$
so therefore $c_2 = 30$ and $s(t) = 8t^2 - \frac{t^4}{12} + 30$.

37. Let $w(t)$ be the number of gallons in the tank at time t.

(a) The water level decreases if $w'(t) = f(t) < 0$ i.e. if $f(t) = 10\sin t < 0$, for which $\pi < t < 2\pi$.
Alternatively, the water level increases if $w'(t) = f(t) > 0$ i.e. if $f(t) = 10\sin t > 0$, for which $0 < t < \pi$.

(b) Now,we start with
$w'(t) = 10\sin t$
Therefore, $\int_0^\pi w'(t)\, dt = \int_0^\pi 10\sin t dt$
$w(\pi) - w(0) = -10\cos t|_0^\pi$

But $w(0) = 100$.
Therefore,
$w(\pi) - 100 = -10(-1-1) = 20$
$\Rightarrow w(\pi) = 120$.
Therefore the tank will have 120 gallons at $t = \pi$.

38. Let $w(t)$ be the number of thousand gallons in the pond at time t.

(a) The water level decreases if $w'(t) = f(t) < 0$ i.e. if $f(t) = 4t - t^2 < 0$, for which $4 < t \le 6$.
Alternatively, the water level increases if $w'(t) = f(t) > 0$ i.e. if $f(t) = 4t - t^2 > 0$, for which $0 < t < 4$.

(b) Now, we start with $w'(t) = 4t - t^2$, Therefore
$$\int_0^6 w'(t)\,dt = \int_0^6 \left(4t - t^2\right) dt$$
$$w(6) - w(0) = \left(2t^2 - \frac{t^3}{3}\right)\Big|_0^6$$
But $w(0) = 40$.
Therefore,
$w(6) - 40 = 72 - 72 = 0$
$\Rightarrow w(6) = 40$.
Therefore the pond has 40,000 gallons at t=6.

39. $y'(x) = \sin\sqrt{x^2 + \pi^2}$.
At the point in question, $y(0) = 0$ and $y'(0) = \sin\pi = 0$. Therefore, the tangent line has slope 0 and passes through the point $(0,\ 0)$. The equation of this line is $y = 0$.

40. $y'(x) = \ln\left(x^2 + 2x + 2\right)$.
At the point in question, $y(-1) = 0$ and $y'(-1) = \ln 1 = 0$. Therefore, the tangent line has slope 0 and passes through the point $(-1,\ 0)$. The equation of this line is $y = 0$.

41. $y'(x) = \cos\left(\pi x^3\right)$.
At the point in question, $y(2) = 0$ and $y'(2) = \cos 8\pi = 1$. Therefore, the tangent line has slope 1 and passes through the point $(2,\ 0)$. The equation of this line is $y = x - 2$.

42. $y'(x) = e^{-x^2+1}$.
At the point in question, $y(0) = 0$ and $y'(0) = e$. Therefore, the tangent line has slope e and passes through the point $(0,\ 0)$. The equation of this line is $y = ex$.

43. $$\int_0^2 \sqrt{x^2+1}dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{\left(\left(\frac{2i}{n}+1\right)\right)}$$
Estimating using $n = 20$, we get the Riemann sum ≈ 2.96.

44. $$\int_0^2 \left(\sqrt{x}+1\right)^2 dx = \int_0^2 \left(x + 2\sqrt{x} + 1\right)dx$$
$$= \left(\frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} + x\right)\Big|_0^2 = 4 + \frac{8\sqrt{2}}{3}.$$

45. $$\int_1^4 \frac{x^2}{x^2+4}dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\frac{\left(1 + (3i/n)^2\right)}{(3i/n)^2 + 4}$$
Estimating using $n = 20$, we get the Riemann sum ≈ 1.71.

46. $$\int_1^4 \frac{x^2+4}{x^2}dx = \int_1^4 1 + \frac{4}{x^2}dx = \left(x - 4x^{-1}\right)\Big|_1^4$$
$= 6$

47. $$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}dx$$
$$= \int_0^{\pi/4} \tan x \sec x dx = \sec x\Big|_0^{\pi/4} = \sqrt{2} - 1$$

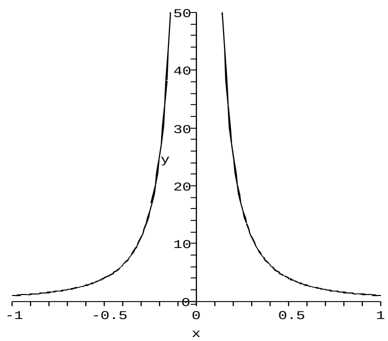
48. $$\int_0^{\pi/4} \frac{\tan x}{\sec^2 x}dx = \int_0^{\pi/4} \sin x \cos x dx$$
$$= \int_0^{\pi/4} \frac{1}{2}\sin 2x dx = \left(-\frac{1}{4}\cos 2x\right)\Big|_0^{\pi/4} = \frac{1}{4}$$

49. From the graph of $f(x)$,
$$\int_0^3 f(x)\,dx < \int_0^2 f(x)\,dx < \int_0^1 f(x)\,dx.$$
The function increases if $g'(x) = f(x) > 0$ i.e. when $x < 1$ or $x > 3$. Thus, the function $g(x)$ is increasing in the intervals $(-\infty, 1)$ and $(3, \infty)$. The function $g(x)$ has critical points at $g'(x) = 0$. i.e. when $x = 1$ or $x = 3$. Therefore the critical points of $g(x)$ are $x = 1$ and $x = 3$.

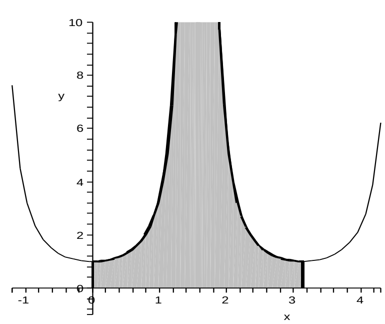
50. $$\int_0^1 f(x)\,dx < \int_0^3 f(x)\,dx < \int_0^2 f(x)\,dx.$$
The function increases if $g'(x) = f(x) > 0$ i.e. when $0 < x < 2$ or $x > 4$. Thus, the function $g(x)$ is increasing in the intervals $(0, 2)$ and $(4, \infty)$. The function $g(x)$ has critical points at $g'(x) = 0$ i.e.when $x = 0$, $x = 2$ and $x = 4$. Therefore the critical points of $g(x)$ are $x = 0$, $x = 2$ and $x = 4$.

51. If you look at the graph of $1/x^2$, it is obvious that there is positive area between the curve and the x-axis over the interval $[-1,1]$. In addition to this, there is a vertical asymptote in the interval that we are integrating over which should alert us to a possible problem.

The problem is that $1/x^2$ is not continuous on $[-1,1]$ (the discontinuity occurs at $x=0$) and that continuity is one of the conditions in the Fundamental Theorem of Calculus, Part I(Theorem 4.1).



52. If you look at the graph of $\sec^2 x$, it is obvious that there is positive area between the curve and the x-axis over the interval $[0,\pi]$. In addition to this, there is a vertical asymptote in the interval that we are integrating over which should alert us to a possible problem. The problem is that $\sec^2 x$ is not continuous on $[0,\pi]$ and that continuity is one of the conditions in the Fundamental Theorem of Calculus, Part I (Theorem 4.1).



53. The integrals in parts (a) and (c) are improper, because the integrands have asymptotes at one of the limits of integration. The Fundamental Theorem of Calculus applies to the integral in part (b).

54. The Fundamental Theorem of Calculus applies to the integral in part (a). The integral in part (b) is improper since the point $x=3$ lies in the interval $[0,4]$, and $\dfrac{1}{(x-3)^2}$ is not defined at $x=3$. The integral in part (c) is improper since the point $x=\pi/2$ lies in the interval $[0,2]$, and $\sec x$ is not defined at $x=\pi/2$.

55. $f_{ave} = \dfrac{1}{3-1}\displaystyle\int_1^3 (x^2-1)dx$
$= \dfrac{1}{2}\left(\dfrac{x^3}{3}-x\right)\Big|_1^3 = \dfrac{10}{3}$

56. $f_{ave} = \dfrac{1}{1-0}\displaystyle\int_0^1 (2x-2x^2)\,dx$
$= \left(x^2-\dfrac{2x^3}{3}\right)\Big|_0^1 = \dfrac{1}{3}$

57. $f_{ave} = \dfrac{1}{\pi/2-0}\displaystyle\int_0^{\pi/2} \cos x dx$
$= \dfrac{2}{\pi}\,(\sin x)|_0^{\pi/2} = \dfrac{2}{\pi}$

58. $f_{ave} = \dfrac{1}{2-0}\displaystyle\int_0^2 e^x dx$
$= \dfrac{1}{2}(e^x)\Big|_0^2$
$= \dfrac{1}{2}\left(e^2-1\right)$

59. (a) Using the Fundamental Theorem of Calculus, it follows that an antiderivative of e^{-x^2} is $\displaystyle\int_a^x e^{-t^2}dt$ where a is a constant.

(b) Using the Fundamental Theorem of Calculus, it follows that an antiderivative of $\sin\sqrt{x^2+1}$ is $\displaystyle\int_a^x \sin\sqrt{t^2+1}dt$ where a is a constant.

60. It may be observed that f is piecewise continuous over its domain.
For $0 < x \le 4$,
$g(x) = \displaystyle\int_0^x f(t)\,dt = \int_0^x \left(t^2+1\right)dt$
$= \left(\dfrac{t^3}{3}+t\right)\Big|_0^x = \dfrac{x^3}{3}+x$
Now, for $x > 4$
$g(x) = \displaystyle\int_0^x f(t)\,dt$
$= \displaystyle\int_0^4 f(t)\,dt + \int_4^x f(t)\,dt$
$= \displaystyle\int_0^4 \left(t^2+1\right)dt + \int_4^x \left(t^3-t\right)dt$
$= \left(\dfrac{t^3}{3}+t\right)\Big|_0^4 + \left(\dfrac{t^4}{4}-\dfrac{t^2}{2}\right)\Big|_4^x$

$$= \left(\frac{4^3}{3}+4\right)+\left(\frac{x^4}{4}-\frac{x^2}{2}-\frac{4^4}{4}+\frac{4^2}{2}\right)$$
$$= \frac{x^4}{4}-\frac{x^2}{2}-\frac{92}{3}$$
$$g(x) = \begin{cases} \frac{x^3}{3}+x & \text{for } 0<x\le 4 \\ \frac{x^4}{4}-\frac{x^2}{2}-\frac{92}{3} & \text{for } 4<x \end{cases}$$

Consider

$$g'(4) = \lim_{h\to 0}\frac{g(4+h)-g(4)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\int_0^{4+h} f(t)\,dt - \int_0^4 f(t)\,dt\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\int_4^{4+h} f(t)\,dt.$$

The Right Hand Limit:

$$\lim_{h\to 0^+}\frac{1}{h}\int_4^{4+h} f(t)\,dt$$
$$= \lim_{h\to 0^+}\frac{1}{h}\int_4^{4+h}\left(t^3-t\right)dt$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{t^4}{4}-\frac{t^2}{2}\right]_4^{4+h}$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{(4+h)^4}{4}-\frac{(4+h)^2}{2}-\frac{4^4}{4}+\frac{4^2}{2}\right]$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{h^4}{4}+4h^3-\frac{47h^2}{2}+60h\right]$$
$$= \lim_{h\to 0^+}\left[\frac{h^3}{4}+4h^2-\frac{47h}{2}+60\right]=60.$$

Now, the Left Hand Limit:

$$\lim_{h\to 0^-}\frac{1}{h}\int_4^{4+h} f(t)\,dt$$
$$= \lim_{h\to 0^-}\frac{1}{h}\int_4^{4+h}\left(t^2+1\right)dt$$
$$= \lim_{h\to 0^-}\frac{1}{h}\left[\frac{t^3}{3}+t\right]_4^{4+h}$$
$$= \lim_{h\to 0^-}\frac{1}{h}\left[\frac{(4+h)^3}{3}+4+h-\frac{4^3}{3}-4\right]$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{h^3+12h^2+48h+64}{3}+h-\frac{64}{3}\right]$$
$$= \lim_{h\to 0^+}\left[\frac{h^2}{3}+4h+17\right]=17.$$

Therefore, $g'(4)$ doesn't exist though $f(4)$ exists. Therefore $g'(x)=f(x)$ is not true for all $x\ge 0$.

61. $f'(x)=x^2-3x+2.$
Setting $f'(x)=0$, we get $(x-1)(x-2)=0$ which implies $x=1, 2$.

$$f'(x) = \begin{cases} >0 & \text{when } t<1 \text{ or } t>2 \\ <0 & \text{when } 1<t<2 \end{cases}$$

$$f(1) = \int_0^1\left(t^2-3t+2\right)dt$$
$$= \left(\frac{t^3}{3}-\frac{3t^2}{2}+2t\right)\Big|_0^1=\frac{5}{6}$$
$$f(2) = \int_0^2\left(t^2-3t+2\right)dt$$
$$= \left(\frac{t^3}{3}-\frac{3t^2}{2}+2t\right)\Big|_0^2=\frac{2}{3}$$

Hence $f(x)$ has a local maximum at the point $\left(1,\frac{5}{6}\right)$ and local minimum at the point $\left(2,\frac{2}{3}\right)$.

62. $g(x)=\int_0^x\left[\int_0^u f(t)\,dt\right]du$
$g'(x)=\int_0^x f(t)\,dt$
$g''(x)=f(x)$
A zero of f corresponds to a zero of the second derivative of g (possibly an inflection point of g).

63. When $a<2$ or $a>2$, f is continuous. Using the Fundamental Theorem of Calculus,

$$\left[\lim_{x\to a}F(x)\right]-F(a)$$
$$= \lim_{x\to a}\left[F(x)-F(a)\right]$$
$$= \lim_{x\to a}\left[\int_0^x f(t)\,dt-\int_0^a f(t)\,dt\right]$$
$$= \lim_{x\to a}\left[\int_a^x f(t)\,dt\right]=0$$

When $a=2$,

$$\lim_{x\to a^-}\left[\int_a^x f(t)\,dt\right]=\lim_{x\to 2^-}\left[\int_2^x t\,dt\right]$$
$$= \lim_{x\to 2^-}\left[\frac{t^2}{2}\right]_2^x=\lim_{x\to 2^-}\left[\frac{x^2}{2}-\frac{2^2}{2}\right]=0$$

and $\lim_{x\to a^+}\left[\int_a^x f(t)\,dt\right]$

$$= \lim_{x\to 2^+}\left[\int_2^x (t+1)\,dt\right]$$
$$= \lim_{x\to 2^+}\left[\frac{t^2}{2}+t\right]_2^x$$
$$= \lim_{x\to 2^+}\left[\frac{x^2}{2}+x-\frac{2^2}{2}-2\right]$$
$$=0$$

Thus, for all value of a,

$$\left[\lim_{x\to a}F(x)\right]-F(a)=0$$
$$\lim_{x\to a}F(x)=F(a)$$

Thus, F is continuous for all x. However, $F'(2)$ does not exist, which is shown as follows:

$$F'(2)=\lim_{h\to 0}\frac{F(2+h)-F(2)}{h}$$

$= \lim_{h\to 0} \frac{1}{h}\left[\int_0^{2+h} f(t)\,dt - \int_0^2 f(t)\,dt\right]$
$= \lim_{h\to 0} \frac{1}{h}\int_2^{2+h} f(t)\,dt$
We will show that this limit does not exist by showing that the left and right limits are different. The right limit is
$\lim_{h\to 0^+} \frac{1}{h}\int_2^{2+h} f(t)\,dt$
$= \lim_{h\to 0^+} \frac{1}{h}\int_2^{2+h} (t+1)\,dt$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{t^2}{2}+t\right]_2^{2+h}$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{(2+h)^2}{2}+2+h-\frac{2^2}{2}-2\right]$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{h^2+4h+4}{2}+2+h-4\right]$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{h^2}{2}+3h\right]$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{h}{2}+3\right]=3$
The left limit is
$\lim_{h\to 0^-} \frac{1}{h}\int_2^{2+h} f(t)\,dt$
$= \lim_{h\to 0^-} \frac{1}{h}\int_2^{2+h} t\,dt$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{t^2}{2}\right]_2^{2+h}$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{(2+h)^2}{2}-\frac{2^2}{2}\right]$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{h^2+4h+4}{2}-2\right]$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{h}{2}+2\right]=2$
Thus, $F'(2)$ does not exist. This result does not contradict the Fundamental Theorem of Calculus, because in this situation, $f(x)$ is not continuous, and thus The Fundamental Theorem of Calculus does not apply.

64. When $x=0$,
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= \lim_{n\to\infty} f(0) = f(0)$
When $0<x<1$,
$\lim_{n\to x^n} = 0$, and then
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= f\left(\lim_{n\to\infty} x^n\right) = f(0)$
$= \lim_{n\to\infty} f(0) = f(0)$
When $x=1$,
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= \lim_{n\to\infty} f(1) = f(1)$.
Thus the integral $\int_0^1 g_n(x)\,dx$ represents the net area between the graph of $f(x^n)$ and the x-axis. As n approaches ∞,
$f(x^n) \to \begin{cases} f(0) & \text{when } 0\le x<1 \\ f(1) & \text{when } x=1 \end{cases}$
Thus the integral $\int_0^1 g_n(x)\,dx$ approaches the area of the shape of a rectangle with length 1 and width $f(0)$ (possibly negative), which means $\lim_{n\to\infty}\int_0^1 g_n(x)\,dx = f(0)$.

65. $\int_0^x [f(t)-g(t)]\,dt$
$= \int_0^x [55+10\cos t-(50+2t)]\,dt$
$= \int_0^x [5+10\cos t-2t]\,dt$
$= 5t+\sin t - t^2\Big|_0^x$
$= 5x+\sin x - x^2$
Since we are integrating the difference in speeds, the integral represents the distance that Katie is ahead at time x. Of course, if this value is negative, is means that Michael is really ahead.

66. (a) $CS = \int_0^Q D(q)\,dq - PQ$
$= \int_0^Q \left(150q-2q-3q^2\right)dq - PQ$
$= \left(150q-q^2-q^3\right)\Big|_0^Q - PQ$
$= 150Q-Q^2-Q^3$
$-\left(150-2Q-3Q^2\right)Q$
$= Q^2+2Q^3$.
When $Q=4$,
$CS = 16+2(64) = 144$ dollors
When $Q=6$, $CS = 36+2(216) = 468$ dollors
The consumer surplus is higher for $Q=6$ than that for $Q=4$.

(b) $CS = \int_0^Q D(q)\,dq - PQ$
$= \int_0^Q 40e^{-0.05q}dq - PQ$
$= \left(-800e^{-0.05q}\right)\Big|_0^Q - PQ$
$= -800e^{-0.05Q}+800-40e^{-0.05Q}$
$= -840e^{-0.05Q}+800$.
When $Q=10$, $CS = -840e^{-0.5}+800 \approx 290.5$ dollors
When $Q=20$, $CS = -840e^{-1}+800 \approx 491.0$ dollors

The consumer surplus is higher for $Q = 20$ than that for $Q = 10$.

67. The next shipment must arrive when the inventory is zero. This occurs at time $T : f(t) = Q - r\sqrt{t}$

$f(T) = 0 = Q - r\sqrt{T}$

$r\sqrt{T} = Q$

$T = \dfrac{Q^2}{r^2}$

The average value of f on $[0, \mathrm{T}]$ is

$$\frac{1}{T}\int_0^T f(t)\,dt$$
$$= \frac{1}{T}\int_0^T \left(Q - rt^{1/2}\right) dt$$
$$= \frac{1}{T}\left[Qt - \frac{2}{3}rt^{3/2}\right]_0^T$$
$$= \frac{1}{T}\left[QT - \frac{2}{3}rT^{3/2}\right]$$
$$= Q - \frac{2}{3}r\sqrt{T}$$
$$= Q - \frac{2}{3}r\frac{Q}{r}$$
$$= \frac{Q}{3}$$

68. The total annual cost $f(Q) = c_0\dfrac{D}{Q} + c_cA = c_0\dfrac{D}{Q} + c_c\dfrac{Q}{3}$

$f'(Q) = -c_0\dfrac{D}{Q^2} + c_c\dfrac{1}{3}$

$f'(Q) = 0$

gives that $Q = \sqrt{\frac{3c_0D}{c_c}}$.

This value of Q minimizes the total cost, since

$$f'(Q)\begin{cases} > 0 & \text{when} \quad Q < \sqrt{\frac{3c_0D}{c_c}} \\ < 0 & \text{when} \quad Q > \sqrt{\frac{3c_0D}{c_c}} \end{cases}$$

When $Q = \sqrt{\dfrac{3c_0D}{c_c}}$,

$$c_0\frac{D}{Q} = c_0\frac{D}{\sqrt{3c_0D/c_c}} = c_c\frac{Q}{3} = c_cA$$

4.6 Integration By Substitution

1. Let $u = x^3 + 2$ and then $du = 3x^2dx$ and
$$\int x^2\sqrt{x^3+2}dx = \frac{1}{3}\int u^{-1/2}du$$
$$= \frac{2}{9}u^{3/2} + c = \frac{2}{9}(x^3+2)u^{3/2} + c.$$

2. Let $u = x^4 + 1$and then $du = 4x^3dx$ and
$$\int x^3(x^4+1)^{-2/3}dx = \frac{1}{4}\int u^{-2/3}du$$
$$= \frac{3}{4}u^{1/3} + c = \frac{3}{4}(x^4+1)^{1/3} + c.$$

3. Let $u = \sqrt{x} + 2$ and then $du = \dfrac{1}{2}x^{-1/2}dx$ and
$$\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}dx = 2\int u^3du$$
$$= \frac{2}{4}u^4 + c = \frac{1}{2}(\sqrt{x}+2)^4 + c.$$

4. Let $u = \sin x$and then $du = \cos x dx$ and
$$\int \sin x\cos x dx = \int u du$$
$$= \frac{u^2}{2} + c = \frac{\sin^2 x}{2} + c.$$

5. Let $u = x^4 + 3$ and then $du = 4x^3dx$ and
$$\int x^3\sqrt{x^4+3}dx = \frac{1}{4}\int u^{1/2}du$$
$$= \frac{1}{6}u^{3/2} + c = \frac{1}{6}(x^4+3)^{3/2} + c.$$

6. Let $u = 1 + 10x$, and then $du = 10dx$ and
$$\int \sqrt{1+10x}dx = \frac{1}{10}\int \sqrt{u}du$$
$$= \frac{1}{10}\int u^{1/2}du = \frac{1}{15}u^{3/2} + c$$
$$= \frac{1}{15}(1+10x)^{3/2} + c.$$

7. Let $u = \cos x$ and then $du = -\sin x dx$ and
$$\int \frac{\sin x}{\sqrt{\cos x}}dx = -\int \frac{du}{\sqrt{u}}$$
$$= -2\sqrt{u} + c = -2\sqrt{\cos x} + c.$$

8. Let $u = \sin x$ and then $du = \cos x dx$ and
$$\int \sin^3 x\cos x dx = \int u^3du$$
$$= \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c.$$

9. Let $u = t^3$ and then $du = 3t^2dt$ and
$$\int t^2\cos t^3dt = \frac{1}{3}\int \cos u du$$
$$= \frac{1}{3}\sin u + c = \frac{1}{3}\sin t^3 + c$$

10. Let $u = \cos t + 3$ and then $du = -\sin t dt$ and
$$\int \sin t(\cos t+3)^{3/4}dt = -\int u^{3/4}du$$
$$= -\frac{4}{7}u^{7/4} + c = -\frac{4}{7}(\cos t+3)^{7/4} + c.$$

11. Let $u = x^2 + 1$ and then $du = 2xdx$ and
$$\int xe^{x^2+1}dx = \int \frac{1}{2}e^u du = \frac{1}{2}e^u + c$$
$$= \frac{1}{2}e^{x^2+1} + c$$

12. Let $u = e^x + 4$ and then $du = e^x dx$ and
$\int e^x\sqrt{e^x+4}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + c$
$= \frac{1}{2}(e^x+4)^{3/2} + c$

13. Let $u = \sqrt{x}$ and then $du = \frac{1}{2\sqrt{x}}dx$ and
$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^u du = 2e^u + c = 2e^{\sqrt{x}} + c$

14. Let $u = \frac{1}{x}$ and then $du = -\frac{1}{x^2}dx$ and
$\int \frac{\cos\left(\frac{1}{x}\right)}{x^2}dx = -\int \cos u du = -\sin u + c$
$= -\sin\frac{1}{x} + c$

15. Let $u = \ln x$ and then $du = \frac{1}{x}dx$ and
$\int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + c$
$= \frac{2}{3}(\ln x)^{3/2} + c$

16. Let $u = \tan x$ and then $du = \sec^2 x dx$ and
Let $u = \ln x$ and then $du = \frac{1}{x}dx$ and
$\int \sec^2 x\sqrt{\tan x}dx = \int u^{1/2}du$
$= \frac{2}{3}u^{3/2} + c = \frac{2}{3}(\sqrt{\tan x})^{3/2} + c$

17. Let $t = \sqrt{u} + 1$ and then
$dt = \frac{1}{2}u^{-1/2}du = \frac{1}{2\sqrt{u}}du$ and
$\int \frac{1}{\sqrt{u}\left(\sqrt{u}+1\right)}du = 2\int \frac{1}{t}dt = 2\ln|t| + c$
$= 2\ln\left|\sqrt{u}+1\right| + c = 2\ln\left(\sqrt{u}+1\right) + c$

18. Let $u = v^2 + 4$ and then $du = 2vdv$ and
$\int \frac{v}{v^2+4}dv = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c$
$= \frac{1}{2}\ln\left|v^2+4\right| + c = \frac{1}{2}\ln\left(v^2+4\right) + c$

19. Let $u = \ln x + 1$ and then $du = \frac{1}{x}dx$ and
$\int \frac{4}{x(\ln x+1)^2}dx = 4\int u^{-2}du$
$= -4u^{-1} + c = -4(\ln x + 1)^{-1} + c$

20. Let $u = \cos 2x$ and then $du = -2\sin 2x dx$ and
$\int \tan 2x dx = -\frac{1}{2}\int \frac{1}{u}du$
$= -\frac{1}{2}\ln|u| + c = -\frac{1}{2}\ln|\cos 2x| + c$

21. Let $u = \sin^{-1}x$ and then $du = \frac{1}{\sqrt{1-x^2}}dx$ and
Let $u = \cos 2x$ and then $du = -2\sin 2x dx$ and
$\int \frac{\left(\sin^{-1}x\right)^3}{\sqrt{1-x^2}}dx = \int u^3 du$
$= \frac{u^4}{4} + c = \frac{\left(\sin^{-1}x\right)^4}{4} + c$

22. Let $u = x^2$ and then $du = 2xdx$ and
$\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$
$= \frac{1}{2}\sin^{-1}u + c = \frac{1}{2}\sin^{-1}x^2 + c$

23. (a) Let $u = x^2$ and then $du = 2xdx$ and
$\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$
$= \frac{1}{2}\sin^{-1}u + c = \frac{1}{2}\sin^{-1}x^2 + c$

(b) Let $u = 1 - x^4$ and then $du = -4x^3dx$ and
$\int \frac{x^3}{\left(1-x^4\right)^{1/2}}dx = -\frac{1}{4}\int u^{-1/2}du$
$= -\frac{1}{2}u^{1/2} + c = -\frac{1}{2}\left(1-x^4\right)^{1/2} + c$

24. (a) Let $u = x^3$ and then $du = 3x^2dx$ and
$\int \frac{x^2}{1+x^6}dx = \frac{1}{3}\int \frac{1}{1+u^2}du$
$= \frac{1}{3}\tan^{-1}u + c = \frac{1}{3}\tan^{-1}x^3 + c$

(b) Let $u = 1 + u^6$ and then $du = 6x^5dx$ and
$\int \frac{x^5}{1+x^6}dx = \frac{1}{6}\int \frac{1}{u}du$
$= \frac{1}{6}\ln|u| + c = \frac{1}{6}\ln\left|1+x^6\right| + c$

25. (a) $\int \frac{1+x}{1+x^2}dx$
$= \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx$
$= \tan^{-1}x + c_1 + \int \frac{x}{1+x^2}dx$
Let $u = 1 + x^2$ and then $du = 2xdx$.
$= \tan^{-1}x + c_1 + \frac{1}{2}\int \frac{1}{u}du$
$= \tan^{-1}x + c_1 + \frac{1}{2}\ln|u| + c_2$
$= \tan^{-1}x + \frac{1}{2}\ln\left|1+x^2\right| + c$
$= \tan^{-1}x + \frac{1}{2}\ln\left(1+x^2\right) + c$

(b) $\int \frac{1+x}{1-x^2}dx = \int \frac{1+x}{(1-x)(1+x)}dx$
$= \int \frac{1}{1-x}dx$
Let $u = 1 - x$ and then $du = -dx$.
$= -\int \frac{1}{u}du = -\ln|u| + c$
$= -\ln|1-x| + c$

26. (a) Let $u = x^{3/2}$ and then
$du = \frac{3}{2}x^{1/2}dx = \frac{3}{2}\sqrt{x}dx$ and
$\int \frac{3\sqrt{x}}{1+x^3}dx = 2\int \frac{1}{1+u^2}du$
$= 2\tan^{-1}u + c = 2\tan^{-1}\left(x^{3/2}\right) + c$

(b) Let $u = x^{\frac{5}{2}}$ and then
$du = \frac{5}{2}x^{\frac{3}{2}}dx = \frac{5}{2}x\sqrt{x}dx$ and
$\int \frac{x\sqrt{x}}{1+x^5}dx = \frac{2}{5}\int \frac{1}{1+u^2}du$
$= \frac{2}{5}\tan^{-1}u + c = \frac{2}{5}\tan^{-1}\left(x^{5/2}\right) + c$

27. Let $u = t+7$ and then $du = dt$, $t = u-7$ and
$\int \frac{2t+3}{t+7}dt = \int \frac{2(u-7)+3}{u}du$
$= \int \left(2 - \frac{11}{u}\right)du = 2u - 11\ln|u| + c$
$= 2(t+7) - 11\ln|t+7| + c$

28. Let $u = t+3$ and then $du = dt$ and
$\int \frac{t^2}{(t+3)^{1/3}}dt = \int \frac{(u-3)^2}{u^{1/3}}du$
$= \int \left(u^{5/3} - 6u^{2/3} + 9u^{-1/3}\right)du$
$= \frac{3}{8}u^{8/3} - \frac{18}{5}u^{5/3} + \frac{18}{2}u^{2/3} + c$
$= \frac{3}{8}(t+3)^{8/3} - \frac{18}{5}(t+3)^{5/3} + \frac{18}{2}(t+3)^{2/3} + c$

29. Let $u = \sqrt{1+\sqrt{x}}$ and then $(u^2-1)^2 = x$,
$2(u^2-1)(2u)du = dx$ and
$\int \frac{1}{\sqrt{1+\sqrt{x}}}dx = \int \frac{4u(u^2-1)}{u}du$
$= 4\int (u^2-1)du = 4(\frac{u^3}{3} - u) + c$
$= \frac{4}{3}(1+\sqrt{x})^{3/2} - 4(1+\sqrt{x})^{1/2} + c$

30. Let $u = x^2$ and then $du = 2xdx$ and
$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{du/2}{u\sqrt{u^2-1}}$
$= \frac{1}{2}\sec^{-1}u + c = \frac{1}{2}\sec^{-1}x^2 + c$

31. Let $u = x^2+1$ and then $u = 2xdx$, $u(0) = 1$,
$u(2) = 5$ and
$\int_0^2 x\sqrt{x^2+1}dx = \frac{1}{2}\int_1^5 \sqrt{u}du$
$= \frac{1}{2}.\frac{2}{3}u^{3/2}\Big|_1^5 = \frac{1}{3}(\sqrt{125}-1) = \frac{5}{3}\sqrt{5} - \frac{1}{3}$

32. Let $u = \pi x^2$ and then $du = 2\pi x dx$ and
$\int_1^3 x\sin(\pi x^2)dx = \frac{1}{2\pi}\int_\pi^{9\pi} \sin u du = (\sin u)\Big|_\pi^{9\pi} =$
0

33. Let $u = t^2+1$ and then $du = 2tdt$,
$u(-1) = 2 = u(1)$ and
$\int_{-1}^1 \frac{t}{(t^2+1)^{1/2}}dt = \frac{1}{2}\int_2^2 u^{-1/2}du = 0$

34. Let $u = t^3$ and then $du = 3t^2dt$,
$u(0) = 0,\ u(2) = 8$ and
$\int_0^2 t^2e^{t^3}dt = \frac{1}{3}\int_0^8 e^u du = \frac{1}{3}e^u\Big|_0^8$
$= \frac{1}{3}\left(e^8 - 1\right)$

35. Let $u = e^x$ and then $du = e^x dx$,
$u(0) = 1$, $u(2) = e^2$ and
$\int_0^2 \frac{e^x}{1+e^{2x}}dx = \int_1^{e^2} \frac{1}{1+u^2}du = \tan^{-1}u\Big|_1^{e^2}$
$= \tan^{-1}e^2 - \frac{\pi}{4}$

36. Let $u = 1+e^x$ and then $du = e^x dx$,
$u(0) = 2$, $u(2) = 1+e^2$ and
$\int_0^2 \frac{e^x}{1+e^x}dx = \int_2^{1+e^2} \frac{1}{u}du = \ln(u)\Big|_2^{1+e^2}$
$= \ln(1+e^2) - \ln(2) = \ln\left(\frac{1+e^2}{2}\right)$

37. Let $u = \sin x$ and then $du = \cos x dx$
$u(\pi/4) = 1/\sqrt{2}, u(\pi/2) = 1$ and
$\int_{\pi/4}^{\pi/2} \cot x dx = \int_{1/\sqrt{2}}^1 \frac{1}{u}du = \ln|u|\Big|_{1/\sqrt{2}}^1$
$= \ln\sqrt{2}$

38. Let $u = \ln x$ and then $du = \frac{1}{x}dx$, $u(1) = 0$,
$u(e) = 1$ and
$\int_1^e \frac{\ln x}{x}dx = \int_0^1 udu = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

39. $\int_1^4 \frac{x-1}{\sqrt{x}}dx = \int_1^4 (x^{1/2} - x^{-1/2})dx$
$= (\frac{2}{3}x^{3/2} - 2x^{1/2})\Big|_1^4$
$= (\frac{16}{3} - 4) - (\frac{2}{3} - 2) = \frac{8}{3}$

40. Let $u = x^2+1$ and then $du = 2xdx$ and
$\int_0^1 \frac{x}{(x^2+1)^{1/2}}dx = \frac{1}{2}\int_1^2 u^{-1/2}du$
$= (u^{1/2})\Big|_1^2 = \sqrt{2} - 1$

41. (a) $\int_0^\pi \sin x^2 dx \approx .77$ using midpoint evaluation with $n \geq 40$.

(b) Let $u = x^2$ and then $du = 2xdx$ and
$$\int_0^{\pi} x \sin x^2 dx = \frac{1}{2}\int_0^{\pi^2} \sin u du$$
$$= \frac{1}{2}(-\cos u)\Big|_0^{\pi^2}$$
$$= -\frac{1}{2}\cos \pi^2 + \frac{1}{2}$$
$$\approx 0.95134$$

42. (a) Let $u = x^2$ and then $du = 2xdx$, $u(-1) = 1$, $u(1) = 1$ and $\int_{-1}^{1} xe^{-x^2} dx = \frac{1}{2}\int_1^1 e^{-u} du = 0$

(b) $\int_{-1}^{1} e^{-x^2} \approx 1.4937$ using midpoint evaluation with $n \geq 50$.

43. (a) $\int_0^2 \frac{4x^2}{(x^2+1)^2} dx \approx 1.414$ using right endpoint evaluation with $n \geq 50$.

(b) Let $u = x^2 + 1$ and then $du = 2xdx$, $x^2 = u - 1$ and
$$\int_0^2 \frac{4x^3}{(x^2+1)^2} dx = \int_1^5 2.\frac{u-1}{u^2} du$$
$$= \int_1^5 (2u^{-1} - 2u^{-2}) du$$
$$= (2\ln|u| + 2u^{-1})\Big|_1^5 = 2\ln 5 - \frac{8}{5}$$

44. (a) $\int_0^{\pi/4} \sec x dx \approx .88$ using midpoint evaluation with $n \geq 10$.

(b) $$\int_0^{\pi/4} \sec^2 x dx = (\tan x)\Big|_0^{\pi/4} = 1.$$

45. $\frac{1}{2}\int_0^4 f(u)du.$

46. $\frac{1}{3}\int_1^8 f(u)du.$

47. $\int_0^1 f(u)du.$

48. $\int_0^4 \frac{f(\sqrt{x})}{\sqrt{x}} dx = 2\int_0^2 f(u)du.$

49. $$\int_{-a}^{a} f(x)dx$$
$$= \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$
Let $u = -x$ and $du = -dx$ in the first integral. Then
$$\int_{-a}^{a} f(x)$$
$$= -\int_{-a}^{0} f(-u)du + \int_0^a f(x)dx$$
$$= \int_0^a f(-u)du + \int_0^a f(x)dx$$
If f is even, then $f(-u) = f(u)$, and so
$$\int_{-a}^{a} f(x)dx$$
$$= \int_0^a f(u)du + \int_0^a f(x)dx$$
$$= \int_0^a f(x)dx + \int_0^a f(x)dx$$
$$= 2\int_0^a f(x)dx$$
If f is odd, then $f(-u) = -f(u)$, and so
$$\int_{-a}^{a} f(x)dx$$
$$= -\int_0^a f(u)du + \int_0^a f(x)dx$$
$$= -\int_0^a f(x)dx + \int_0^a f(x)dx$$
$$= 0$$

50. First, let $u = x - T$, then for any a,
$$\int_T^{a+T} f(x)dx = \int_0^a f(u+T)du$$
$$= \int_0^a f(u)du = \int_0^a f(x)dx$$
If we let $a = T$, then we get
$$\int_a^T f(x)dx = \int_T^{2T} f(x)dx.$$
If we let $a = 2T$, then we get
$$\int_0^{2T} f(x)dx = \int_T^{3T} f(x)dx$$
and then
$$\int_0^T f(x)dx = \int_T^{2T} f(x)dx$$
$$= \int_0^{2T} f(x)dx - \int_0^T f(x)dx$$
$$= \int_T^{3T} f(x)dx - \int_T^{2T} f(x)dx$$
$$= \int_{2T}^{3T} f(x)dx$$
It is straight forward to see that for any integer i,
$$\int_0^T f(x)dx = \int_{iT}^{(i+1)T} f(x)dx$$
Now suppose $0 \leq a \leq T$, then
$$\int_0^T f(x)dx - \int_a^{a+T} dx$$

$= \int_0^a f(x)dx - \int_T^{a+T} f(x)dx$

So $\int_0^T f(x)dx = \int_a^{a+T} dx$

Now suppose a is any number. Then a must lie in some interval $[iT, (i+1)T]$ for some interger i. Use the similar method as in above, we shall get

$\int_{iT}^{(i+1)T} f(x)dx = \int_a^{a+T} f(x)dx$

And since $\int_{iT}^{(i+1)T} f(x)dx = \int_0^T f(x)dx$,

we get $\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$

51. (a) Let $u = 10 - x$, so that $du = -dx$. Then,

$I = \int_0^{10} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{10-x}} dx$

$= -\int_{x=0}^{x=10} \frac{\sqrt{10-u}}{\sqrt{10-u} + \sqrt{u}} du$

$= -\int_{u=10}^{u=0} \frac{\sqrt{10-u}}{\sqrt{10-u} + \sqrt{u}} du$

$= \int_{u=0}^{u=10} \frac{\sqrt{10-u}}{\sqrt{10-u} + \sqrt{u}} du$

$I = \int_{x=0}^{x=10} \frac{\sqrt{10-x}}{\sqrt{10-x} + \sqrt{x}} dx$

The last equation follows from the previous one because u and x are dummy variables of integration. Now note that

$\frac{\sqrt{x}}{\sqrt{x} + \sqrt{10-x}}$

$= \frac{\sqrt{x} + \sqrt{10-x} - \sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}$

$= 1 - \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}$

Thus,

$\int_0^{10} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{10-x}} dx$

$= \int_0^{10} \left[1 - \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}\right] dx$

$= \int_0^{10} 1dx - \int_0^{10} \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}} dx$

$I = \int_0^{10} 1dx - I$

$2I = 10$

$I = 5$

(b) Let $u = a - x$, so that $du = -dx$ Then,

$I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx$

$= -\int_a^0 \frac{f(a-u)}{f(a-u) + f(u)} du$

$= \int_0^a \frac{f(a-u)}{f(a-u) + f(u)} du$

$I = \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} dx$

The last equation follows from the previous one because u and x are dummy variables of integration. Now note that

$\frac{f(x)}{f(x) + f(a-x)}$

$= \frac{f(x) + f(a-x) - f(a-x)}{f(x) + f(a-x)}$

$= 1 - \frac{f(a-x)}{f(a-x) + f(x)}$

Thus,

$\int_0^a \frac{f(x)}{f(x) + f(a-x)} dx$

$= \int_0^a \left[1 - \frac{f(a-x)}{f(a-x) + f(x)}\right] dx$

$= \int_0^a 1dx - \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} dx$

$2I = a$

$I = a/2$

52. (a) Let $u = 6 - x$, so that $du = -dx$. Then,

$I = \int_2^4 \frac{\sin^2(9-x)}{\sin^2(9-x) + \sin^2(x+3)} dx$

$= -\int_4^2 \frac{\sin^2(u+3)}{\sin^2(u+3) + \sin^2(9-u)} du$

$= \int_2^4 \frac{\sin^2(u+3)}{\sin^2(u+3) + \sin^2(9-u)} du$

$= \int_2^4 \frac{\sin^2(x+3)}{\sin^2(x+3) + \sin^2(9-x)} dx$

$= \int_2^4 \left[1 - \frac{\sin^2(9-x)}{\sin^2(x+3) + \sin^2(9-x)}\right] dx$

$I = \int_2^4 1dx - I$

$2I = 2$

$I = 1$

(b) Let $u = 6 - x$, so that $du = -dx$. Then,

$I = \int_2^4 \frac{f(9-x)}{f(9-x) + f(x+3)} dx$

$= -\int_4^2 \frac{f(u+3)}{f(u+3) + f(9-u)} du$

$= \int_2^4 \frac{f(u+3)}{f(u+3) + f(9-u)} du$

$= \int_2^4 \frac{f(x+3)}{f(x+3) + f(9-x)} dx$

$$= \int_2^4 \left[1 - \frac{f(9-x)}{f(x+3)+f(9-x)}\right] dx$$
$$I = \int_2^4 1dx - I$$
$$2I = 2$$
$$I = 1$$

53. Let $6-u = x+4$; that is, let $u = 2-x$, so that $du = -dx$.
Then,
$$I = \int_0^2 \frac{f(x+4)}{f(x+4)+f(6-x)} dx$$
$$= -\int_2^0 \frac{f(6-u)}{f(6-u)+f(u+4)} du$$
$$= \int_0^2 \frac{f(6-u)}{f(6-u)+f(u+4)} du$$
$$= \int_0^2 \frac{f(6-x)}{f(6-x)+f(x+4)} dx$$
$$= \int_0^2 \frac{f(6-x)+f(x+4)-f(x+4)}{f(6-x)+f(x+4)} dx$$
$$= \int_0^2 \left[1 - \frac{f(x+4)}{f(6-x)+f(x+4)}\right] dx$$
$$I = \int_0^2 1dx - I$$
$$2I = 2$$
$$I = 1$$

54. (a) Let $u = x^{1/6}$, so that $du = \frac{1}{6}x^{-5/6}dx$.
Then,
$$I = \int \frac{1}{x^{5/6}+x^{2/3}} dx$$
$$= \int \frac{x^{-5/6}dx}{1+x^{-1/6}}$$
$$= \int \frac{6\,du}{1+\frac{1}{u}}$$
$$= \int \frac{6u}{u+1}\, du$$
Let $v = u+1$, then $dv = du$ and $u = v-1$.
Then, $I = \displaystyle\int \frac{6u}{u+1}\, du$
$$= \int \frac{6(v-1)}{v}\, dv$$
$$= \int \left(6 - \frac{6}{v}\right) dv$$
$$= 6v - 6\ln|v| + c$$
$$= 6(u+1) - 6\ln|u+1| + c$$
$$= 6(x^{1/6}+1) - 6\ln|x^{1/6}+1| + c$$

(b) Let $u = x^{1/6}$, so that $du = (1/6)x^{-5/6}dx$, which means $6u^5du = dx$.
Thus,
$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}} dx$$
$$= 6\int \frac{u^5}{u^3+u^2} du$$
$$= 6\int \frac{u^3}{u+1} du$$
$$= 6\int \left[u^2 - u + 1 - \frac{1}{u+1}\right] du$$
$$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + c$$
$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6}$$
$$= -6\ln|x^{1/6}+1| + c$$

(c) Let $u = x^{1/q}$, then $q\,du = x^{(1-q)/q}dx$, and
$$I = \int \frac{1}{x^{(p+1)/q}+x^{p/q}} dx$$
$$= \int \frac{x^{(1-q)/q}dx}{x^{(p+2-q)/q}+x^{(p+1-q)/q}} dx$$
$$= q\int \frac{1}{u^{p+2-q}+u^{p+1-q}} du$$
$$= q\int \frac{u^{q-1-p}}{u+1} du$$
The rest of the calculation will depend on the values of p and q.

55. First let $u = \ln\sqrt{x}$, so that $du = x^{-1/2}(1/2)x^{-1/2}dx$, so that $2du = \dfrac{1}{x}dx$. Then,
$$\int \frac{1}{x\ln\sqrt{x}} dx = 2\int \frac{1}{u} du$$
$$= 2\ln|u| + c$$
$$= 2\ln|\ln\sqrt{x}| + c$$
Now use the substitution $u = \ln x$, so that $du = \frac{1}{x}dx$. Then,
$$\int \frac{1}{x\ln\sqrt{x}} dx = \int \frac{1}{x\ln(x^{1/2})} dx$$
$$= \int \frac{1}{x\left(\frac{1}{2}\right)\ln x} dx$$
$$= 2\int \frac{1}{u} du$$
$$= 2\ln|u| + c_1$$
$$= 2\ln|\ln x| + c_1$$
The two results differ by a constant, and so are equivalent, as can be seen as follows:
$$2\ln|\ln\sqrt{x}| = 2\ln|\ln(x^{1/2})|$$
$$= 2\ln\left|\frac{1}{2}\ln x\right|$$
$$= 2\left[\ln\frac{1}{2} + \ln|\ln x|\right]$$
$$= 2\ln\frac{1}{2} + 2\ln|\ln x|$$
$$= 2\ln|\ln x| + \text{ constant}$$

56. The area of the region bounded by the curve $y = \pi x - x^2$ and x-axis, where $0 \le x \le 1$ is
$$\int_0^1 \left(\pi x - x^2\right) dx$$

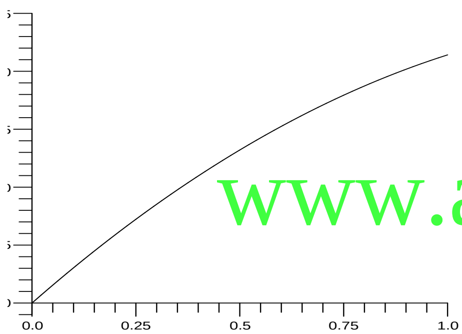
$= \left(\pi\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$
$= \frac{\pi}{2} - \frac{1}{3}.$

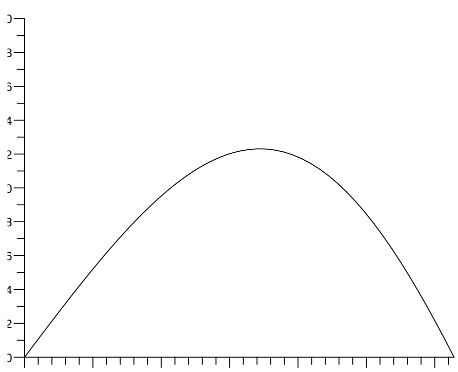
The area of the region bounded by the curve $y = \left(\pi\cos x - \cos^2 x\right)\sin x$ and x-axis, where $0 \le x \le \frac{\pi}{2}$ is $\int_0^{\pi/2}\left(\pi\cos x - \cos^2 x\right)\sin x dx$.

Let $u = \cos x$ and then $du = -\sin x dx$.

$u(0) = 1, u\left(\frac{\pi}{2}\right) = 0.$
$= \int_1^0 \left(-\pi u + u^2\right) du$
$= -\pi\left(\frac{u^2}{2}\right) + \frac{u^3}{3}\Big|_1^0$
$= \frac{\pi}{2} - \frac{1}{3}$

Thus, the areas are equal.





57. The point is that if we let $u = x^4$, then we get $x = \pm u^{1/4}$, and so we need to pay attention to the sign of u and x. A safe way is to solve the original indefinite integral in terms of x, and then solve the definite integral using boundary points in terms of x.

$\int_{-2}^{1} 4x^4 dx = \int_{x=-2}^{x=1} u^{1/4} du$
$= \frac{4}{5}u^{5/4}\Big|_{x=-2}^{x=1} = \frac{4}{5}x^5\Big|_{x=-2}^{x=1}$
$= \frac{4}{5}\left(1^5 - (-2)^5\right) = \frac{4}{5}(1-(-32)) = \frac{132}{5}$

58. The problem is that it is not true on entire interval $[0,\pi]$ that $\cos x = \sqrt{1-\sin^2 x}$. This is only true on the interval $\left[0, \frac{\pi}{2}\right]$. To make this substitution correctly, one must break up the integral:

$\int_0^{\pi} \cos x(\cos x)dx$
$= \int_0^{\pi/2} \cos x(\cos x)dx + \int_{\pi/2}^{\pi} \cos x(\cos x)dx$
$= \int_{x=0}^{x=\pi/2} \sqrt{1-u^2}du$
$- \int_{x=\pi/2}^{x=\pi} \sqrt{1-u^2}du$
$= \left(\frac{u}{2} + \frac{\sin^{-1}u}{2}\right)\Big|_{x=0}^{x=\pi/2}$
$- \left(\frac{u}{2} + \frac{\sin^{-1}u}{2}\right)\Big|_{x=\pi/2}^{x=\pi}$
$= \left(\frac{\sin x}{2} + \frac{\sin^{-1}(\sin x)}{2}\right)\Big|_{x=0}^{x=\pi/2}$
$- \left(\frac{\sin x}{2} + \frac{\sin^{-1}(\sin x)}{2}\right)\Big|_{x=\pi/2}^{x=\pi}$
$= \left(\frac{1}{2} + \frac{\pi}{4}\right) - 0 - 0 + \left(\frac{1}{2} + \frac{\pi}{4}\right)$
$= 1 + \frac{\pi}{2}$

59. Let $u = 1/x$, so that $du = -1/x^2 dx$, which means that $-1/u^2 du = dx$. Then,

$\int_0^1 \frac{1}{x^2+1}dx = -\int_{1/a}^{1} \frac{1/u^2}{1/u^2+1}du$
$= \int_1^{1/a} \frac{1}{1+u^2}du = \int_1^{1/a} \frac{1}{1+x^2}dx$

The last equation follows from the previous one because u and x are dummy variables of integration. Thus,

$\tan^{-1}x\big|_a^1 = \tan^{-1}x\big|_1^{1/a}$
$\tan^{-1}1 - \tan^{-1}a = \tan^{-1}\frac{1}{a} - \tan^{-1}1$
$\tan^{-1}a + \tan^{-1}\frac{1}{a} = 2\tan^{-1}1$
$\tan^{-1}a + \tan^{-1}\frac{1}{a} = \frac{\pi}{2}$

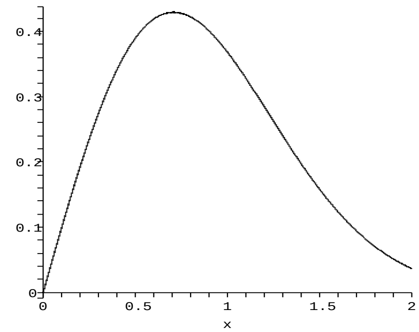
60. If $u = 1/x$, then $du = -dx/x^2$ and

$\int \frac{1}{|x|\sqrt{x^2-1}}dx$
$= \int \frac{1}{x^2\sqrt{x^2-1}}dx$
$= -\int \frac{1}{\sqrt{1-u^2}}du$
$= -\sin^{-1}u + c$
$= -\sin^{-1}1/x + c$

On the other hand,
$\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1}x + c_1$
So $-\sin^{-1}1/x = \sec^{-1}x + c_2$.
Let $x = 1$, we get
$\sin^{-1}1 = \sec^{-1}1 + c_2$
$\frac{\pi}{2} = 0 + c_2$
$c_2 = \frac{\pi}{2}$

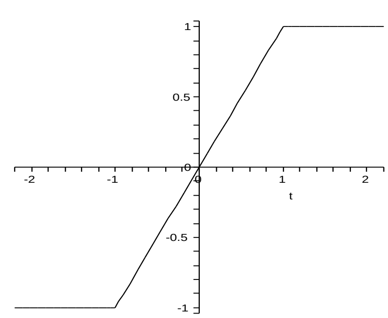
61. $\overline{x} = \frac{\int_{-2}^{2} x\sqrt{4-x^2}dx}{\int_{-2}^{2}\sqrt{4-x^2}dx}$
Examine the denominator of $\overline{x}$, the graph of $\sqrt{4-x^2}$, which is indeed a semicircle, is symmetric over the two intervals $[-2,0]$ and $[0,2]$, while multiplying by x changes the symmetry into anti-symmetry. In other words,
$\int_{-2}^{0} x\sqrt{4-x^2}dx = -\int_{0}^{2} x\sqrt{4-x^2}dx$
so that
$\int_{-2}^{2} x\sqrt{4-x^2}dx$
$= \int_{-2}^{0} x\sqrt{4-x^2}dx + \int_{0}^{2} x\sqrt{4-x^2}dx = 0$
Hence $\overline{x} = 0$.
Now the integral $\int_{-2}^{2}\sqrt{4-x^2}dx$ is the area of a semicircle with radius 2, thus its value is $(1/2)\pi 2^2 = 2\pi$. Then
$\overline{y} = \frac{\int_{-2}^{2}\left(\sqrt{4-x^2}\right)^2 dx}{2\int_{-2}^{2}\sqrt{4-x^2}dx}$
$= \frac{\int_{-2}^{2}\left(4-x^2\right)dx}{2.2\pi}$
$= \frac{\int_{-2}^{0}\left(4-x^2\right)dx + \int_{0}^{2}\left(4-x^2\right)dx}{4\pi}$
$= \frac{2\int_{0}^{2}\left(4-x^2\right)dx}{4\pi}$
$= \frac{\int_{0}^{2}\left(4-x^2\right)dx}{2\pi}$
$= \frac{1}{2\pi}\left(4x - \frac{x^3}{3}\right)\Big|_0^2 = \frac{8}{3\pi}$

62. These animals are likely to be found 0.7 miles from the pond. Let $u = -x^2$, then $du = -2xdx$, $u(0) = 0$, $u(2) = -4$ and
$\int_0^2 xe^{-x^2}dx = -\frac{1}{2}\int_0^{-4} e^u du$
$= -\frac{1}{2}\left(e^{-4} - 1\right) = \frac{1 - e^{-4}}{2}$



63. $V(t) = V_p\sin(2\pi ft)V^2(t)$
$= V_p^2\sin^2(2\pi ft)$
$= V_p^2\left(\frac{1}{2} - \frac{1}{2}\cos(4\pi ft)\right)$
$= \frac{V_p^2}{2}\left(1 - \cos(4\pi ft)\right)$
$\text{rms} = \sqrt{f\int_0^{1/f} V^2(t)\,dt}$
$= \sqrt{f\int_0^{1/f}\frac{V_p^2}{2}\left(1-\cos(4\pi ft)\right)dt}$
$= \frac{V_p\sqrt{f}}{\sqrt{2}}\sqrt{\left(t - \frac{\sin(4\pi ft)}{4\pi f}\right)\Big|_0^{1/f}}$
$= \frac{V_p\sqrt{f}}{\sqrt{2}}\sqrt{\frac{1}{f}} = \frac{V_p}{\sqrt{2}}$

64. $\int_{-2}^{2} f^2(t)dt$
$= \int_{-2}^{-1} 1dt + \int_{-1}^{1} t^2 dt + \int_1^2 1dt$
$= 1 + \frac{2}{3} + 1 = \frac{8}{3}$
$\text{rms} = \sqrt{\frac{1}{4}\int_1^2 f^2(t)\,dt}$
$= \sqrt{\frac{1}{4}\left(\frac{8}{3}\right)} = \sqrt{\frac{2}{3}}$



4.7 Numerical Integration

1. Midpoint Rule:

$\int_0^1 (x^2+1)\,dx$
$\approx \frac{1}{4}\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right]$
$=\frac{85}{64}$

Trapezoidal Rule:

$\int_0^1 (x^2+1)\,dx$
$\approx \frac{1-0}{2(4)}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right.$
$\left.+f(1)\right]$
$=\frac{43}{32}$

Simpson's Rule:

$\int_0^1 (x^2+1)\,dx$
$=\frac{1-0}{3(4)}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)\right.$
$\left.+f(1)\right]$
$=\frac{4}{3}$

2. Midpoint Rule:

$\int_0^2 (x^2+1)\,dx$
$\approx \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)\right]$
$=\frac{1}{2}\left(\frac{17}{16}+\frac{25}{16}+\frac{41}{16}+\frac{65}{16}\right)$
$=\frac{37}{8}$

Trapezoidal Rule:

$\int_0^2 (x^2+1)\,dx$
$\approx \frac{1}{4}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)\right.$
$\left.+f(2)\right]$
$=\frac{1}{4}\left(1+\frac{5}{2}+4+\frac{13}{2}+5\right)$
$=\frac{19}{4}$

Simpson's Rule:

$\int_0^2 (x^2+1)\,dx$
$=\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)\right.$
$\left.+f(2)\right]$
$=\frac{1}{6}(1+5+4+13+5)$
$=\frac{14}{3}$

3. Midpoint Rule:

$\int_1^3 \frac{1}{x}dx$
$\approx \frac{3-1}{4}\left[f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)\right]$
$=\frac{1}{2}\left(\frac{4}{5}+\frac{4}{7}+\frac{4}{9}+\frac{4}{11}\right)$
$=\frac{3776}{3465}$

Trapezoidal Rule:

$\int_1^3 \frac{1}{x}dx$
$\approx \frac{3-1}{2(4)}\left[f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)\right.$
$\left.+f(3)\right]$
$=\frac{1}{4}\left(1+\frac{4}{3}+1+\frac{4}{5}+\frac{1}{3}\right)$
$=\frac{67}{60}$

Simpson's Rule:

$\int_1^3 \frac{1}{x}dx$
$=\frac{3-1}{3(4)}\left[f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)\right.$
$\left.+f(3)\right]$
$=\frac{1}{6}\left(1+\frac{8}{3}+1+\frac{8}{5}+\frac{1}{3}\right)$
$=\frac{11}{10}$

4. Midpoint Rule:

$\int_{-1}^1 (2x-x^2)\,dx$
$\approx \frac{1}{2}\left[f\left(-\frac{3}{4}\right)+f\left(-\frac{1}{4}\right)+f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right]$
$=\frac{1}{2}\left(-\frac{33}{16}-\frac{9}{16}+\frac{7}{16}+\frac{15}{16}\right)$
$=\frac{-5}{8}$

Trapezoidal Rule:

$\int_{-1}^1 (2x-x^2)\,dx$
$\approx \frac{1}{4}\left[f(-1)+2f\left(-\frac{1}{2}\right)+2f(0)+2f\left(\frac{1}{2}\right)\right.$
$\left.+f(1)\right]$

$= \frac{1}{4}\left(-3-\frac{5}{2}+0+\frac{3}{2}+1\right)$
$= -\frac{3}{4}$

Simpson's Rule:

$\int_{-1}^{1}\left(2x-x^2\right)dx$
$\approx \frac{1}{6}\left[f(-1)+4f\left(-\frac{1}{2}\right)+2f(0)+4f\left(\frac{1}{2}\right)\right.$
$\left.+f(1)\right]$
$= \frac{1}{6}(-3-5+0+3+1)$
$= -\frac{2}{3}$

5. Midpoint Rule:

$\ln 4 - 1.366162 = 1.386294 - 1.366162$
$= 0.020132$

Trapezoidal Rule:

$\ln 4 - 1.428091 = 1.386294 - 1.428091$
$= -0.041797$

Simpson's Rule:

$\ln 4 - 1.391621 = 1.386294 - 1.391621$
$= -0.005327$

Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

6. Midpoint Rule:
$\ln 8 - 1.987287 = 2.079442 - 1.987287$
$= 0.092155$
Trapezoidal Rule:
$\ln 8 - 2.289628 = 2.079442 - 2.289628$
$= -0.210186$
Simpson's Rule:
$\ln 8 - 2.137327 = 2.079442 - 2.137327$
$= -0.057885$
Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

7. Midpoint Rule:
$\sin 1 - 0.843666 = 0.841471 - 0.843666$
$= -0.002195$
Trapezoidal Rule: $\sin 1 - 0.837084 = 0.841471 - 0.837084$
$= 0.004387$
Simpson's Rule:
$\sin 1 - 0.841489 = 0.841471 - 0.841489$
$= -0.000018$
Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

8. Midpoint Rule: $e^2 - 7.322986 = 7.389056 - 7.322986$
$= 0.06607$
Trapezoidal Rule: $e^2 - 7.52161 = 7.389056 - 7.52161$
$= -0.132554$
Simpson's Rule: $e^2 - 7.391210 = 7.389056 - 7.391210$
$= -0.002154$
Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

9. $\int_0^{\pi}\cos x^2 dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.5538	0.5889	0.5660
20	0.5629	0.5713	0.5655
50	0.5652	0.566	0.5657

10. $\int_0^{\frac{\pi}{4}}\sin \pi x^2 dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.386939	0.385578	0.386476
20	0.386600	0.386259	0.386485
50	0.386504	0.386450	0.386486

11. $\int_0^{2} e^{-x^2}dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.88220	0.88184	0.88207
20	0.88211	0.88202	0.88208
50	0.88209	0.88207	0.88208

12. $\int_0^{3} e^{-x^2}dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.886210	0.886202	0.886207
20	0.886208	0.886206	0.886207
50	0.886207	0.886207	0.886207

13. $\int_0^{\pi} e^{\cos x}dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	3.9775	3.9775	3.9775
20	3.9775	3.9775	3.9775
50	3.9775	3.9775	3.9775

14. $\int_0^{1}\sqrt[3]{x^2+1}dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	3.333017	3.336997	3.334337
20	3.334012	3.335007	3.334344
50	3.334291	3.334450	3.334344

15. The exact value of this integral is

$$\int_0^1 5x^4 dx = x^5\big|_0^1 = 1 - 0 = 1$$

n	*Midpoint*	EM_n
10	1.00832	8.3×10^{-3}
20	1.00208	2.1×10^{-3}
40	1.00052	5.2×10^{-3}
80	1.00013	1.3×10^{-3}

n	*Trapezoidal*	ET_n
10	0.98335	1.6×10^{-2}
20	0.99583	4.1×10^{-3}
40	0.99869	1.0×10^{-3}
80	0.99974	2.6×10^{-4}

n	*Simpson*	ES_n
10	1.000066	6.6×10^{-5}
20	1.0000041	4.2×10^{-6}
40	1.00000026	2.6×10^{-7}
80	1.00000016	1.6×10^{-8}

16. The exact value of this integral is

$$\int_1^2 \frac{1}{x} dx = \ln 2$$

n	*Midpoint*	EM_n
10	0.692835	3.1×10^{-4}
20	0.693069	7.8×10^{-5}
40	0.693128	2.0×10^{-5}
80	0.693142	4.9×10^{-6}

n	*Trapezoidal*	ET_n
10	0.693771	6.2×10^{-4}
20	0.693303	1.6×10^{-4}
40	0.693186	3.9×10^{-5}
80	0.693157	9.8×10^{-6}

n	Simpson	ES_n
10	0.693150	3.1×10^{-6}
20	0.693147	1.9×10^{-7}
40	0.693147	1.2×10^{-8}
80	0.693147	8.0×10^{-10}

17. The exact value of this integral is

$$\int_0^\pi \cos x dx = \sin x|_0^\pi = 0$$

n	*Midpoint*	EM_n
10	0	0
20	0	0
40	0	0
80	0	0

n	*Trapezoidal*	ET_n
10	0	0
20	0	0
40	0	0
80	0	0

n	*Simpson*	ES_n
10	0	0
20	0	0
40	0	0
80	0	0

18. The exact value of this integral is

$$\int_0^{\frac{\pi}{4}} \cos x dx = \frac{1}{\sqrt{2}}$$

n	*Midpoint*	EM_n
10	0.707289	1.8×10^{-4}
20	0.707152	4.5×10^{-5}
40	0.707118	1.1×10^{-5}
80	0.707110	2.8×10^{-6}

n	*Trapezoidal*	ET_n
10	0.706743	3.6×10^{-4}
20	0.707016	9.1×10^{-5}
40	0.707084	2.3×10^{-5}
80	0.707101	5.7×10^{-6}

n	*Simpson*	ES_n
10	0.7071087	1.5×10^{-7}
20	0.7071068	9.5×10^{-9}
40	0.7071068	6×10^{-10}
80	0.7071068	6×10^{-10}

19. If you double the error in the Midpoint Rule is divided by 4, the error in the Trapezoidal Rule is divided by 4 and the error in the Simpson's Rule is divided by 16.

20. If you halve the interval length $b - a$ the error in the Midpoint Rule is divided by 8, the error in the Trapezoidal Rule is divided by 8 and the error in the Simpson's Rule is divided by 32.

21. Trapezoidal Rule:

$$\int_0^2 f(x)\,dx$$
$$\approx \frac{2-0}{2(8)}[f(0) + 2f(0.25) + 2f(0.5)$$
$$+ 2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5)$$
$$+ 2f(1.75) + f(2)]$$
$$= \frac{1}{8}[4.0 + 9.2 + 10.4 + 9.6 + 10 + 9.2 + 8.8$$
$$+ 7.6 + 4.0]$$
$$= 9.1$$

Simpson's Rule:

$$\int_0^2 f(x)\,dx$$
$$\approx \frac{2-0}{3(8)}[f(0) + 4f(0.25) + 2f(0.5)$$
$$+ 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5)$$
$$+4f(1.75) + f(2)]$$

$= \frac{1}{12}[4.0 + 18.4 + 10.4 + 19.2 + 10.0$
$+18.4 + 8.8 + 15.2 + 4.0]$
≈ 9.033

22. Trapezoidal Rule:
$\int_0^2 f(x)\,dx$
$\approx \frac{0.25}{2}[f(0) + 2f(0.25) + 2f(0.5)$
$+2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5)$
$+2f(1.75) + f(2)]$
$= \frac{0.25}{2}[(1.0) + 2(0.6) + 2(0.2) + 2(-0.2)$
$+ 2(-0.4) + 2(0.4) + 2(0.8)$
$+ 2(1.2) + (2.0)]$
$= 1.025.$
Simpson's Rule:
$\int_0^2 f(x)\,dx$
$\approx \frac{0.25}{3}[f(0) + 4f(0.25) + 2f(0.5)$
$+ 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5)$
$+4f(1.75) + f(2)]$
$= \frac{0.25}{3}[(1.0) + 4(0.6) + 2(0.2) + 4(-0.2)$
$+2(-0.4) + 4(0.4) + 2(0.8) + 4(1.2) + (2.0)]$
≈ 1.016667

23. (a) $f(x) = \frac{1}{x}, f''(x) = \frac{2}{x^3}, f^{(4)}(x) = \frac{24}{x^5}$.
Then $K = 2$, $L = 24$. Hence according to Theorems 9.1 and 9.2,

$|ET_4| \le 2\frac{(4-1)^3}{12 \cdot 4^2} \approx 0.281$

$|EM_4| \le 2\frac{(4-1)^3}{24 \cdot 4^2} \approx 0.141$

$|ES_4| \le 24\frac{(4-1)^5}{180 \cdot 4^2} \approx 0.127$

(b) Using Theorems 9.1 and 9.2, and the calculation in Example 9.10, we find the following lower bounds for the number of steps needed to guarantee accuracy of 10^{-7} in Exercise 5:

Midpoint: $\sqrt{\frac{2 \cdot 3^3}{24 \cdot 10^{-7}}} \approx 4745$

Trapezoidal: $\sqrt{\frac{2 \cdot 3^3}{14 \cdot 10^{-7}}} \approx 6709$

Simpson's: $\sqrt[4]{\frac{24 \cdot 3^5}{180 \cdot 10^{-7}}} \approx 135$

24. (a) $f(x) = \cos x, f''(x) = -\cos x$,
$f^{(4)}(x) = \cos x$.Then $K = L = 1$.
Hence according to
Theorems 9.1 and 9.2,

$|ET_4| \le 1\frac{1}{12 \cdot 4^2} \approx 0.005$

$|EM_4| \le 1\frac{1}{24 \cdot 4^2} \approx 0.003$

$|ES_4| \le 1\frac{1}{180 \cdot 4^4} \approx 2.17 \times 10^{-5}$

(b) Midpoint: $|E_n|K\frac{(b-a)^3}{24n^2} = \frac{1}{24n^2}$

We want $\frac{1}{24n^2} \le 10^7$

$24n^2 \ge 10^7$

$n^2 \ge \frac{10^7}{24}$

$n \ge \sqrt{\frac{10^7}{24}} \approx 645.5$

So need $n \ge 646$.

Trapezoid: $|ET_n|K\frac{(b-a)^3}{12n^2} = \frac{1}{12n^2}$

We want $n^2 \ge \frac{10^7}{12}$

$n \ge \sqrt{\frac{10^7}{12}} \approx 912.87$

$n \ge 913$

Simpson: $|ES_n|L\frac{(b-a)^5}{180n^4} = \frac{1}{180n^4}$

$\frac{1}{180n^4} \le 10^{-7}$
$180n^4 \ge 10^7$

$n^4 \ge \frac{10^7}{180}$

$n \ge \sqrt[4]{\frac{10^7}{180}} \approx 15.4$

So need $n \ge 16$.

25. (a) $f(x) = \ln x$. Hence, $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$. Therefore $|f''(x)| \le 1$.
The error using Trapezoidal Rule is

$|E(T_n)| \le 1\frac{(2-1)^3}{12n^2} \le 10^{-6}$

$|E(T_n)| \le \frac{1}{12n^2} \le 10^{-6}$
Solving for n,
$|E(T_n)| \le \frac{1}{12}10^6 \le n^2$

$n \ge \sqrt{\frac{1}{12}10^6}$
≈ 288.67

(b) $f(x) = \ln x$. Hence, $f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}$. Therefore $|f''(x)| \le 1$.
The error using Midpoint Rule is

$|E(M_n)| \le 1\dfrac{(2-1)^3}{24n^2} \le 10^{-6}$
$|E(M_n)| \le \dfrac{1}{24n^2} \le 10^{-6}$
Solving for n,
$|E(M_n)| \le \dfrac{1}{24}10^6 \le n^2$
$n \ge \sqrt{\dfrac{1}{24}10^6}$
≈ 204.12

(c) $f(x) = \ln x$. Hence, $f'(x) = \dfrac{1}{x}$, $f''(x) = -\dfrac{1}{x^2}$, $f'''(x) = \dfrac{2}{x^3}$ and $f^{(4)}(x) = -\dfrac{6}{x^4}$. Therefore $\left|f^{(4)}(x)\right| \le 6$.
The error using Simpson's Rule is
$|E(S_n)| \le 6\dfrac{(2-1)^4}{180n^4} \le 10^{-6}$
$|E(S_n)| \le \dfrac{1}{30n^4} \le 10^{-6}$
Solving for n,
$|E(S_n)| \le \dfrac{1}{30}10^6 \le n^4$
$n \ge \sqrt[4]{\dfrac{1}{30}10^6}$
≈ 13.5

26. (a) $f(x) = x\ln x$. Hence, $f'(x) = 1+\ln x$ and $f''(x) = \dfrac{1}{x}$. Therefore $|f''(x)| \le 1$.
$|E(T_n)| \le 1\dfrac{(4-1)^3}{12n^2} \le 10^{-6}$
$|E(T_n)| \le \dfrac{27}{12n^2} \le 10^{-6}$
Solving for n,
$|E(T_n)| \le \dfrac{27}{12}10^6 \le n^2$
$n \ge \sqrt{\dfrac{27}{12}10^6}$
$= 1500.$

(b) $f(x) = x\ln x$. Hence, $f'(x) = 1+\ln x$, $f''(x) = \dfrac{1}{x}$. Therefore $|f''(x)| \le 1$.
The error using Trapezoidal Rule is
$|E(M_n)| \le 1\dfrac{(4-1)^3}{24n^2} \le 10^{-6}$
$|E(M_n)| \le \dfrac{27}{24n^2} \le 10^{-6}$
Solving for n,
$|E(M_n)| \le \dfrac{27}{24}10^6 \le n^2$
$n \ge \sqrt{\dfrac{27}{24}10^6}$
≈ 1060.66

(c) $f(x) = x\ln x$. Hence, $f'(x) = 1+\ln x$, $f''(x) = \dfrac{1}{x}$, $f'''(x) = -\dfrac{1}{x^2}$ and $f^{(4)}(x) = \dfrac{2}{x^3}$.
Therefore $\left|f^{(4)}(x)\right| \le 2$.
The error using Simpson's Rule is
$|E(S_n)| \le 2\dfrac{(4-1)^4}{180n^4} \le 10^{-6}$
$|E(S_n)| \le \dfrac{9}{10n^4} \le 10^{-6}$
Solving for n,
$|E(S_n)| \le \dfrac{9}{10}10^6 \le n^4$
$n \ge \sqrt[4]{\dfrac{9}{10}10^6}$
≈ 30.8

27. (a) $f(x) = e^{x^2}$. Hence, $f'(x) = 2xe^{x^2}$, $f''(x) = 2e^{x^2}\left(2x^2+1\right)$. Therefore, $|f''(x)| \le 6e \approx 16.3097$.
The error using Trapezoidal Rule is
$|E(T_n)| \le 16.3097\dfrac{(1-0)^3}{12n^2} \le 10^{-6}$
$|E(T_n)| \le \dfrac{16.3097}{12n^2} \le 10^{-6}$
Solving for n,
$|E(T_n)| \le \dfrac{16.3097}{12}10^6 \le n^2$
$n \ge \sqrt{\dfrac{16.3097}{12}10^6}$
$\approx 1165.$

(b) $f(x) = e^{x^2}$. Hence, $f'(x) = 2xe^{x^2}$, $f''(x) = 2e^{x^2}\left(2x^2+1\right)$. Therefore, $|f''(x)| \le 6e \approx 16.3097$.
The error using Trapezoidal Rule is
$|E(M_n)| \le 16.3097\dfrac{(1-0)^3}{24n^2} \le 10^{-6}$
$|E(M_n)| \le \dfrac{16.3097}{24n^2} \le 10^{-6}$
Solving for n,
$|E(M_n)| \le \dfrac{16.3097}{24}10^6 \le n^2$
$n \ge \sqrt{\dfrac{16.3097}{24}10^6}$
≈ 824.36

(c) $f(x) = e^{x^2}$. Hence,
$f'(x) = 2xe^{x^2}$,
$f''(x) = 2e^{x^2}\left(2x^2+1\right)$,
$f'''(x) = 4e^{x^2}\left(2x^3+3x\right)$
$f^{(4)}(x) = 4e^{x^2}\left(4x^4+12x^2+3\right)$.

Therefore, $|f''(x)| \le 76e \approx 206.5823$.
The error using Simpson's Rule is

$|E(S_n)| \le 206.5823\dfrac{(1-0)^4}{180n^4} \le 10^{-6}$

$|E(S_n)| \le \dfrac{206.5823}{180n^2} \le 10^{-6}$

Solving for n,

$|E(S_n)| \le \dfrac{206.5823}{180}10^6 \le n^2$

$n \ge \sqrt[4]{\dfrac{206.5823}{180}10^6}$

≈ 32.7307.

28. (a) $f(x) = xe^x$

Hence,

$f'(x) = e^x(x+1)$

$f''(x) = e^x(x+2)$

Therefore,

$|f''(x)| \le 4e^2 \approx 21.21$

The error using Midpoint Rule is

$|E(M_n)| \le 21.21\dfrac{(2-1)^3}{24n^2} \le 10^{-6}$

$|E(M_n)| \le \dfrac{21.21}{24n^2} \le 10^{-6}$

Solving for n,

$|E(M_n)| \le \dfrac{2402.0293}{24}10^6 \le n^2$

$n \ge \sqrt{\dfrac{21.21}{24}10^6}$

≈ 940.0797838

(b) $f(x) = xe^x$

Hence,

$f'(x) = e^x(x+1)$

$f''(x) = e^x(x+2)$

Therefore,

$|f''(x)| \le 4e^2 \approx 21.21$

The error using Trapezoidal Rule is

$|E(T_n)| \le 21.21\dfrac{(2-1)^3}{12n^2} \le 10^{-6}$

$|E(T_n)| \le \dfrac{21.21}{12n^2} \le 10^{-6}$

Solving for n,

$|E(T_n)| \le \dfrac{21.21}{12}10^6 \le n^2$

$n \ge \sqrt{\dfrac{21.21}{12}10^6}$

≈ 1329.473580

(c) $f(x) = xe^x$

Hence,

$f'(x) = e^x(x+1), f''(x) = e^x(x+2)$

$f'''(x) = e^x(x+3)$

$f^{(4)}(x) = e^x(x+4)$

Therefore,

$\left|f^{(4)}(x)\right| \le 6e^2 \approx 31.82$

The error using Simpson's Rule is

$|E(S_n)| \le 31.82\dfrac{(2-1)^4}{180n^4} \le 10^{-6}$

$|E(S_n)| \le \dfrac{31.82}{180n^2} \le 10^{-6}$

Solving for n,

$|E(S_n)| \le \dfrac{31.82}{180}10^6 \le n^2$

$n \ge \sqrt[4]{\dfrac{31.82}{180}10^6}$

≈ 20.50486515

29. We use $K = 60$, $L = 120$

n	EM_n	*Error Bound*
10	8.3×10^{-3}	2.5×10^{-2}

n	ET_n	*Error Bound*
10	1.6×10^{-2}	5×10^{-2}

n	ES_n	*Error Bound*
10	7.0×10^{-5}	6.6×10^{-3}

30. We use $K = L = 1$.

n	EM_n	*Error Bound*
10	0	1.3×10^{-2}

n	ET_n	*Error Bound*
10	0	2.6×10^{-2}

n	ES_n	*Error Bound*
10	0	1.7×10^{-4}

31. (a) Left Endpoints:

$\displaystyle\int_0^2 f(x)dx$

$\approx \dfrac{2-0}{4}[f(0) + f(.5) + f(1)$

$+f(1.5)]$

$= \dfrac{1}{2}(1 + .25 + 0 + .25)$

$= .75$

(b) Midpoint Rule:

$\displaystyle\int_0^2 f(x)dx$

$\approx \dfrac{2-0}{4}[f(.25) + f(.75)$

$+f(1.25) + f(1.75)]$

$= \dfrac{1}{2}(.65 + .15 + .15 + .65)$

$= .7$

(c) Trapezoidal Rule:

$\displaystyle\int_0^2 f(x)dx$

$\approx \frac{2-0}{2(4)}\left[f(0)+2f(.5)+2f(1)\right.$
$\left.+2f(1.5)+f(2)\right]$
$= \frac{1}{4}(1+.5+0+.5+1)$
$= .75$

(d) Simpson's rule: $\int_0^2 f(x)\,dx$
$= \frac{2}{12}\left[f(0)+4f(0.5)+2f(1)\right.$
$\left.+4f(1.5)+f(2)\right]$
$= \frac{1}{6}\left[1+4(0.25)+2(0)+4(0.25)+1\right]$
$= \frac{1}{6}[4]$
$= 0.66666$

32. (a) Left Endpoints:
$\int_0^2 f(x)dx$
$\approx \frac{1}{2}\left(f(0)+f(.5)+f(1)+f(1.5)\right)$
$= \frac{1}{2}(0.5+0.8+0.5+0.1)$
$= 0.95$

(b) Midpoint Rule:
$\int_0^2 f(x)\,dx$
$\approx \frac{1}{2}(0.7+0.8+0.4+0.2)$
$= 1.05$

(c) Trapezoidal Rule:
$\int_0^2 f(x)dx$
$\approx \frac{1}{4}[0.5+2(0.8)+2(0.5)+2(0.1)$
$+0.5]$
$= 0.95$

(d) Simpson's rule: $\int_0^2 f(x)\,dx$
$= \frac{2-0}{12}\left[f(0)+4f(0.5)+2f(1)\right.$
$\left.+4f(1.5)+f(2)\right]$
$= \frac{1}{6}\left[0.5+4(0.9)+2(0.5)+4(0.1)+0.5\right]$
$= \frac{1}{6}[0.5+3.6+1+0.4+0.5]$
$= 1$

33. (a) Midpoint Rule:
$M_n < \int_a^b f(x)dx$

(b) Trapezoidal Rule:
$T_n > \int_a^b f(x)dx$

(c) Simpson's Rule:
Not enough information.

34. (a) Midpoint Rule:
$M_n < \int_a^b f(x)dx$

(b) Trapezoidal Rule:
$T_n > \int_a^b f(x)dx$

(c) Simpson's Rule:
$S_n \geq \int_a^b f(x)dx$

35. (a) Midpoint Rule:
$M_n > \int_a^b f(x)dx$

(b) Trapezoidal Rule:
$T_n < \int_a^b f(x)dx$

(c) Simpson's Rule:
Not enough information.

36. (a) Midpoint Rule: $M_n > \int_a^b f(x)dx$

(b) Trapezoidal Rule: $T_n < \int_a^b f(x)dx$

(c) Simpson's Rule: $S_n \leq \int_a^b f(x)dx$

37. (a) Midpoint Rule: $M_n < \int_a^b f(x)dx$

(b) Trapezoidal Rule: $T_n > \int_a^b f(x)dx$

(c) Simpson's Rule: $S_n = \int_a^b f(x)dx$

38. (a) Midpoint Rule: $M_n = \int_a^b f(x)dx$

(b) Trapezoidal Rule: $T_n = \int_a^b f(x)dx$

(c) Simpson's Rule: $S_n = \int_a^b f(x)dx$

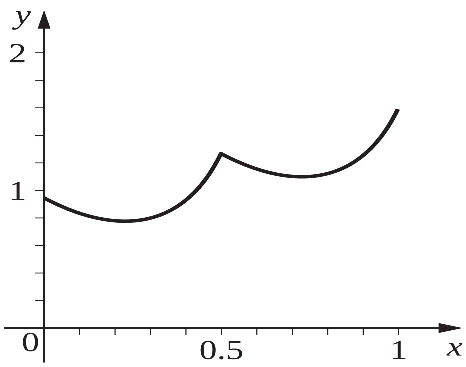
39. $\frac{1}{2}(R_L + R_R)$

$= \sum_{i=0}^{n-1} f(x_i) + \sum_{i=1}^{n} f(x_i)$

$= f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \sum_{i=1}^{n-1} f(x_i) + f(x_n)$

$= f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) = T_n$

40.



41. $I_1 = \int_0^1 \sqrt{1-x^2}dx$ is one fourth of the area of a circle with radius 1, so $\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$

$I_2 = \int_0^1 \frac{1}{1+x^2}dx = \arctan x|_0^1$

$= \arctan 1 - \arctan 0 = \frac{\pi}{4}$

n	$S_n(\sqrt{1-x^2})$	$S_n(\frac{1}{1+x^2})$
4	0.65652	0.78539
8	0.66307	0.78539

The second integral $\int \frac{1}{1+x^2}dx$ provides a better algorithm for estimating π.

42. $\int_{-h}^{h} (Ax^2 + Bx + c)\, dx$

$= \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + cx\right)\Big|_{-h}^{h}$

$= \frac{2}{3}Ah^3 + 2Ch$

$= \frac{h}{3}(2Ah^2 + 6C)$

$= \frac{h}{3}[f(-h) + 4f(0) + f(h)]$

43. (a) $\int_{-1}^{1} x dx = 0$

$\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right) = 0$

(b) $\int_{-1}^{1} x^2 dx = \frac{2}{3}$

$\left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^3 = \frac{2}{3}$

(c) $\int_{-1}^{1} x^3 dx = 0$

$\left(-\frac{1}{\sqrt{3}}\right)^3 + \left(\frac{1}{\sqrt{3}}\right)^3 = 0$

44. Simpson's Rule with $n = 2$:

$\int_{-1}^{1} \pi \cos\left(\frac{\pi x}{2}\right) dx$

$\approx \frac{2}{6}\left(f(-1) + 4f\left(\frac{-1}{3}\right) + f(1)\right)$

$= \frac{1}{3}\left[\pi \cos\left(\frac{-\pi}{2}\right) + 4\pi \cos\left(\frac{-\pi}{6}\right) + \pi \cos\left(\frac{\pi}{2}\right)\right]$

$= \frac{\pi}{3}\left(0 + 2\sqrt{3} + 0\right) = \frac{2\pi}{\sqrt{3}}$

≈ 3.6276

Gaussian quadrature:

$\int_{-1}^{1} \pi \cos\left(\frac{\pi x}{2}\right) dx$

$\approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$= \pi \cos\left(-\frac{\pi}{2\sqrt{3}}\right) + \pi \cos\left(\frac{\pi}{2\sqrt{3}}\right)$

≈ 3.87164

45. Simpson's Rule is not applicable because $\frac{\sin x}{x}$ is not defined at $x = 0$. $L = \lim_{x\to 0} \frac{\sin x}{x}$

$= \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$

The two functions $f(x)$ and $\frac{\sin x}{x}$ differ only at one point,so $\int_0^{\pi} f(x)\, dx = \int_0^{\pi} \frac{\sin x}{x} dx$ We can now apply Simpson's Rule with $n = 2$:

$\int_0^{\pi} f(x)\, dx$

$\approx \frac{\pi}{6}\left(1 + \frac{4\sin \pi}{\frac{\pi}{2}} + \frac{\sin \pi}{\pi}\right)$

$= \frac{\pi}{2}\left(\frac{1}{3} + \frac{8}{3\pi}\right)$

$\approx \dfrac{\pi}{2} \cdot 1.18$

46. The function $\dfrac{\sin x}{x}$ is not defined at $x = 0$, and it is symmetric across the y-axis. We define a new function

$$f(x) = \begin{cases} \sin x / x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

over the interval $[0, \pi/2]$, and $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x} dx =$
$2 \displaystyle\int_0^{\pi/2} f(x) dx$

Use Simpson's Rule on $n = 2$:

$$\int_0^{\pi/2} f(x) dx$$
$$\approx \frac{\pi}{12} \left(1 + \frac{\frac{\sqrt{2}}{2}}{\pi/4} + \frac{1}{\pi/2} \right)$$
$$\approx \frac{\pi}{2} \cdot 15.22$$

Hence

$$\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x} dx \approx \frac{\pi}{2} \cdot 30.44$$

47. Let I be the exact integral. Then we have

$T_n - I \approx -2(M_n - I)$
$T_n - I \approx 2I - 2M_n$
$T_n + 2M_n \approx 3I$
$\dfrac{T_n}{3} + \dfrac{2}{3} M_n \approx I$

48. The text does not say this, but we want to show that

$\frac{1}{3} T_n + \frac{2}{3} M_n = S_{2n}$

In this case, we have data points: $x_0, x_1, x_2, x_3, ..., x_{2n}$.
The midpoint rule will use the points: $x_1, x_3, ..., x_{2n-1}$.
The trapezoidal rule will use the points: $x_0 x_2, ..., x_{2n}$.

$$\frac{1}{3} T_n + \frac{2}{3} M_n$$
$$= \left(\frac{1}{3}\right) \left(\frac{b-a}{2n}\right) [f(x_0) + 2f(x_2) + 2f(x_4)$$
$$+ ... + 2f(x_{2n-2}) + f(x_{2n})]$$
$$+ \left(\frac{2}{3}\right) \left(\frac{b-a}{n}\right) \times [f(x_1) + f(x_3)$$
$$+ f(x_5) + ... + f(x_{2n-1}) + f(x_{2n})]$$
$$= \left(\frac{b-a}{2n}\right) [f(x_0) + 4f(x_1) + 2f(x_2)$$
$$+ 4f(x_3) + 2f(x_4) + ... + 2f(x_{2n-2})$$
$$+ 4f(x_{2n-1}) + f(x_{2n})]$$
$$= S_{2n}$$

49. $f(x) + f(1-x)$

$$= \frac{x^2}{2x^2 - 2x + 1} + \frac{(1-x)^2}{2(1-x)^2 - 2(1-x) + 1}$$
$$= \frac{x^2}{2x^2 - 2x + 1}$$
$$+ \frac{(1-x)^2}{2(1 - 2x + x^2) - 2 + 2x + 1}$$
$$= \frac{x^2}{2x^2 - 2x + 1} + \frac{(1-x)^2}{2x^2 - 2x + 1}$$
$$= \frac{x^2}{x^2 + (x-1)^2} + \frac{(1-x)^2}{(1-x)^2 + x^2}$$
$$= \frac{x^2 + (1-x)^2}{x^2 + (1-x)^2}$$
$$= 1$$

By Trapezoidal Rule,

$$\int_0^1 f(x)\, dx$$
$$= \frac{(1-0)}{2n} [f(x_0) + 2f(x_1)$$
$$+ 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)]$$
$$= \frac{(1-0)}{2n} \left[f(0) + 2f(\frac{1}{n}) \right.$$
$$\left. + 2f(\frac{2}{n}) + ... + 2f(\frac{n-1}{n}) + f(1) \right]$$

as $f(x) + f(1-x) = 1$,
we have,

$f(0) + f(1) = 1$,
$f(\frac{1}{n}) + f(\frac{n-1}{n}) = 1$
$f(\frac{2}{n}) + f(\frac{n-2}{n}) = 1$
.
.
.
$f(\frac{n-1}{n}) + f(\frac{1}{n}) = 1$

Adding the above n equations, we get

$$\left[f(0) + 2f(\frac{1}{n}) + .. + 2f(\frac{n-1}{n}) + f(1) \right] = n$$

Hence,

$$\int_0^1 f(x)\, dx = \frac{1}{2n}(n) = \frac{1}{2}$$

50. $\displaystyle\int_0^n x^n dx$

$$= \left(\frac{n-0}{2n}\right) [f_0 + 2f_1 + 2f_2 + ... + 2f_{n-1} + f_n]$$
$$= \frac{1}{2} [f(0) + 2f(1) + 2f(2) + 2f(3) + ...$$
$$+ 2f(n-1) + f(n)]$$
$$= \frac{1}{2} [(n^n) + 2(1^n + 2^n + 3^n + ... + (n-1)^n]$$

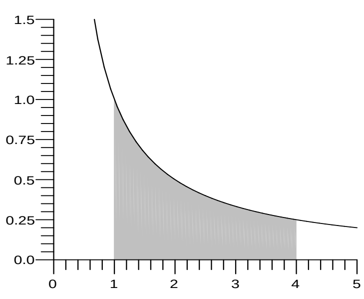
Now
$$\int_0^n x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^n$$
$$= \frac{n^{n+1}}{n+1}$$

The sum of the areas of the trapezoids is greater than the area defined by the curve over the interval 0 to n. $\frac{n^{n+1}}{n+1} < \frac{n^n}{2} + 1^n + 2^n + 3^n + ... + (n-1)^n$

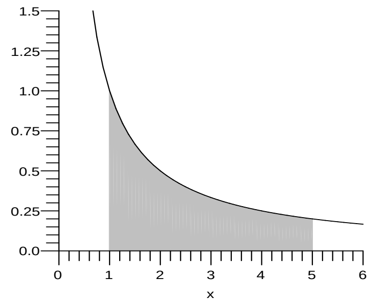
$$\frac{n^{n+1}}{n+1} + \frac{n^n}{2} < 1^n + 2^n + +... + (n-1)^n + n^n$$
$$\frac{2n^{n+1} + n^{n+1} + n^n}{2(n+1)} < 1^n + 2^n + ... + n^n$$
$$\frac{3n^{n+1} + n^n}{2(n+1)} < 1^n + 2^n + 3^n + ... + n^n$$
$$\frac{(3n+1)}{2(n+1)} n^n < 1^n + 2^n + 3^n + ... + n^n$$

4.8 The Natural Logarithm As An Integral

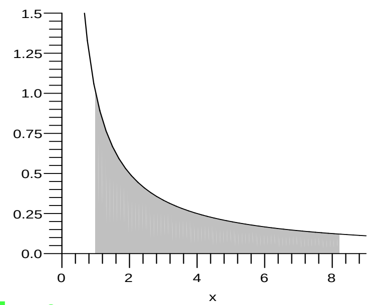
1. $\ln 4 = \ln 4 - \ln 1 = \ln x|_1^4 = \int_1^4 \frac{dx}{x}$



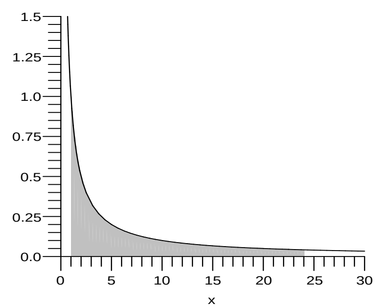
2. $\ln 5 = \int_1^5 \frac{dx}{x}$



3. $\ln 8.2 = \int_1^{8.2} \frac{dx}{x}$



4. $\ln 24 = \int_1^{24} \frac{dx}{x}$



5. $\ln 4 = \int_1^4 \frac{dx}{x}$
$$\approx \frac{3}{12}\left(\frac{1}{1} + 4\frac{1}{1.75} + 2\frac{1}{1.5} + 4\frac{1}{3.25} + \frac{1}{4}\right)$$
$$\approx 1.3868$$

6. $\ln 5 = \int_1^5 \frac{dx}{x}$
$$\approx \frac{4}{12}\left(\frac{1}{1} + 4\frac{1}{2} + 2\frac{1}{3} + 4\frac{1}{4} + \frac{1}{5}\right)$$
$$\approx 1.6108$$

7. (a) Simpson's Rule with $n = 32$:

$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.386296874$

(b) Simpson's Rule with $n = 64$:
$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.386294521$

8. (a) Simpson's Rule with $n = 32$:
$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.609445754$

(b) Simpson's Rule with $n = 64$:
$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.609438416$

9. $\frac{7}{2} \ln 2$

10. $\ln 2$

11. $\ln \left(\frac{3^2 \cdot \sqrt{3}}{9} \right) = \frac{1}{2} \ln 3$

12. $\ln \left(\frac{\frac{1}{9} \cdot \frac{1}{9}}{3} \right) = -5 \ln 3$

13. $\frac{1}{\sqrt{x^2+1}} \cdot \frac{1}{2} \left(x^2+1\right)^{-\frac{1}{2}} .2x$

14. $\frac{5x^4 \sin x \cos x + x^5 \cos^2 x - x^5 \sin x}{x^5 \sin x \cos x}$

15. $\frac{x^5+1}{x^4} \cdot \frac{4x^3 \left(x^5+1\right) - x^4 \left(5x^4\right)}{\left(x^5+1\right)^2}$

16. $\sqrt{\frac{x^5+1}{x^3}} \cdot \frac{1}{2} \cdot \left(\frac{x^3}{x^5+1} \right)^{-1/2}$
$\cdot \frac{3x^2 \left(x^5+1\right) - x^3 \left(5x^4\right)}{\left(x^5+1\right)^2}$

17. $\frac{d}{dx} \frac{1}{2} \left(\frac{\ln \left(x^2+1\right)}{\ln 7} \right)$
$= \frac{1}{2 \ln 7} \frac{d}{dx} \left(\ln \left(x^2+1\right) \right)$
$= \frac{1}{\ln 7} \left(\frac{x}{x^2+1} \right)$

18. $\frac{d}{dx} \left(\frac{x \ln 2}{\ln 10} \right) = \frac{\ln 2}{\ln 10} \frac{d}{dx} (x) = \log_{10} 2$

19. Let $y = 3^{\sin x}$
On taking natural logarithm.
$\ln y = \ln \left(3^{\sin x}\right) = \sin x \ln 3$
$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left(\sin x \ln 3\right) = \ln 3 \frac{d}{dx} \left(\sin x\right)$
$\frac{1}{y} \frac{dy}{dx} = (\ln 3) \cos x$
$\frac{dy}{dx} = y \left(\ln 3\right) \cos x$
$\frac{dy}{dx} = 3^{\sin x} \left(\ln 3\right) \cos x$

20. $y = 4^{\sqrt{x}}$
On taking natural logarithm.
$\ln y = \ln \left(4^{\sqrt{x}}\right) = \sqrt{x} \ln 4$
$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{x} \ln 4\right)$
$= (\ln 4) \frac{d}{dx} \left(\sqrt{x}\right)$
$= (\ln 4) \left(\frac{1}{2\sqrt{x}} \right)$
$\frac{dy}{dx} = y \frac{(\ln 4)}{2\sqrt{x}}$
$\frac{dy}{dx} = \frac{4^{\sqrt{x}} (\ln 4)}{2\sqrt{x}}$

21. $\int \frac{1}{x \ln x} dx = \ln |\ln x| + c$

22. $\int \frac{1}{\sqrt{1-x^2} \sin^{-1} x} dx = \ln \left|\sin^{-1} x\right| + c$

23. Let $u = x^2, du = 2x dx$
$\int x 3^{x^2} dx = \frac{1}{2} \int 3^u du = \frac{3^{x^2}}{2 \ln 3} + c$

24. Let $u = 2^x,\ du = 2^x (\ln 2) dx$
$\int 2^x \sin \left(2^x\right) dx = \frac{1}{\ln 2} \int \sin (u)\, du$
$= \frac{-\cos \left(2^x\right)}{\ln 2} + c$

25. Let $u = \frac{2}{x},\ du = \left(\frac{-2}{x^2} \right) dx$
$\int \frac{e^{2/x}}{x^2} dx = -\frac{1}{2} \int e^u du$
$= -\frac{1}{2e^u} + c = -\frac{1}{2} e^{2/x} + c$

26. Let $u = \ln x^3, du = \left(\frac{3}{x} \right) dx$
$\int \frac{\sin \left(\ln x^3\right)}{x} dx = \frac{1}{3} \int \sin u du$
$= -\frac{1}{3} \cos u + c$
$= -\frac{1}{3} \cos \left(\ln x^3\right) + c$

27. $\int_0^1 \frac{x^2}{x^3-4} dx$
$= \frac{1}{3} \ln \left|x^3-4\right|_0^1$
$= \frac{1}{3} \ln 3 - \frac{1}{3} \ln 4 = \frac{1}{3} \ln \frac{3}{4}$

28. $\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$
$= \ln \left|e^x + e^{-x}\right|\Big|_0^1$

$= \ln\left(e + e^{-1}\right) - \ln 2$
$= \ln\left(\dfrac{e + e^{-1}}{2}\right)$

29. $\int_0^1 \tan x dx = \int_0^1 \dfrac{\sin x}{\cos x} dx$
$= -\ln |\cos x||_0^1$
$= -\ln|\cos 1| - \ln|\cos 0|$
$= -\ln(\cos 1)$

30. Let $u = \ln x$, $du = \dfrac{dx}{x}$
$\int \dfrac{\ln x}{x} dx = \int u dx = \dfrac{u^2}{2} + c$
$= \dfrac{(\ln x)^2}{2} + c$
$\int_1^2 \dfrac{\ln x}{x} dx = \left.\dfrac{(\ln x)^2}{2}\right|_1^2$
$= \dfrac{\ln^2 2}{2} - \dfrac{\ln^2 1}{2} = \dfrac{\ln^2 2}{2}$

31. $\ln\left(\dfrac{a}{b}\right) = \ln\left(a \cdot \dfrac{1}{b}\right) = \ln a + \ln\left(\dfrac{1}{b}\right)$
$= \ln a - \ln b$

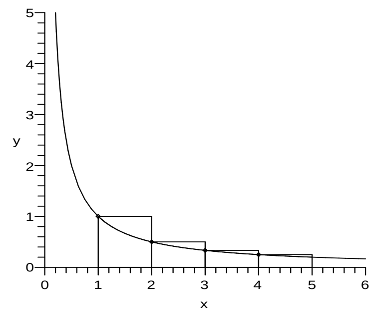
32. Consider $x = 2^{-n}$, where n is any integer for $x > 0$.

On taking natural logarithm.

$\ln x = \ln 2^{-n}$
$\Rightarrow \ln x = -n \ln 2$
Now $x \to 0, 2^{-n} \to 0 \Rightarrow n \to \infty$
$\Rightarrow \lim\limits_{x\to 0_+} (\ln x) = \lim\limits_{n\to\infty} (-n \ln 2)$
$= -(\ln 2) \lim\limits_{n\to\infty} (n)$.
But, $\ln 2 \approx 0.6931$ and $\lim\limits_{n\to\infty} n = \infty$
$\Rightarrow \lim\limits_{x\to 0_+} (\ln x) = -\infty$.

33. We know that by definition, $\ln(n) = \int_1^n \dfrac{1}{x} dx$

which is the area bounded by the curve $y = \dfrac{1}{x}$, the positive x-axis between the ordinates $x = 1$ and $x = n$. Let $y = f(x) = \dfrac{1}{x}$.



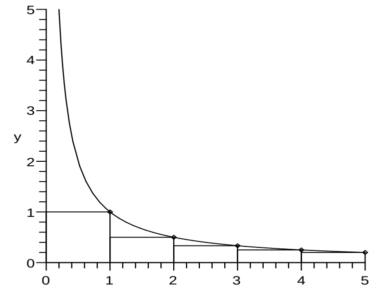
From the graph, it may be observed that the area bounded by $y = \dfrac{1}{x}$; the x-axis between the ordinates $x = 1$ and $x = n$ is lesser than the shaded area which is the sum of areas of the $(n-1)$ rectangles having width 1 unit and height $f(i)$
Thus from the graph,

$\int_1^n \dfrac{1}{x} dx < \sum\limits_{i=1}^{n-1} (f(i) \times 1)$
$\ln(n) < f(1) + f(2) + f(3) + ...$
$.. + f(n-1)$
or $\ln(n) < 1 + \dfrac{1}{2} + ... + \dfrac{1}{n-1}$
Hence proved. We know that,
$\lim\limits_{n\to\infty} \ln(n) = \infty$
$\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{2} + ... + \dfrac{1}{n-1}\right) \geq \lim\limits_{n\to\infty} \ln(n)$
$= \infty$

34. We know that by definition,
$\ln(n) = \int_1^n \dfrac{1}{x} dx$
which is the area bounded by the curve

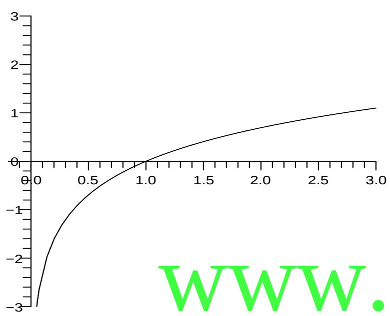
$y = \dfrac{1}{x}$, the positive x-axis between the ordnates $x = 1$ and $x = n$.

Let $y = f(x) = \dfrac{1}{x}$.



Let us consider $(n-1)$ rectangles, having width 1 unit and height $f(i+1)$ where $i = 1, 2, 3, \ldots\ldots, n-1$. Thus from the graph,
$\int_1^n \frac{1}{x} dx > \sum_{i=1}^{n-1} (f(i+1) \times 1)$
$\ln(n) > f(2) + f(3) + \ldots.. + f(n)$
or $\ln(n) > \frac{1}{2} + \frac{1}{3} \ldots.. + \frac{1}{n}$.

35. Since the domain of the function $y = \ln x$ is $(0, \infty)$, $f'(x) = \frac{1}{x} > 0$ for $x > 0$. So f is increasing throughout the domain. Similarly, $f''(x) = -\frac{1}{x^2} < 0$ for $x > 0$. Therefore, the graph is concave down everywhere, the graph of the function $y = \ln x$ is as below.



36. Proof of (ii)

By using the rules of logarithm we have,
$\ln\left(\frac{e^r}{e^s}\right) = \ln(e^r) - \ln(e^s)$
$= r \ln e - s \ln e = r - s = \ln\left(e^{r-s}\right)$
Since $\ln x$ is one to one, it follows that
$\frac{e^r}{e^s} = e^{r-s}$.
Proof of (iii)

By using the rules of logarithm we have,
$\ln(e^r)^t = t \ln(e^r) = rt \ln e = \ln\left(e^{rt}\right)$
Since $\ln x$ is one to one, it follows that
$(e^r)^t = e^{rt}$.

37. $h = \ln e^h = \int_1^{e^h} \frac{1}{x} dx = \frac{e^h - 1}{\bar{x}}$,
for some $\bar{x}$ in $(0, h)$
$\frac{e^h - 1}{h} = \bar{x}$
as $h \to 0^+, \bar{x} \to 0$, then
$\lim_{h \to 0^+} \frac{e^h - 1}{h} = 0$
$-h = \ln e^{-h} = \int_1^{e^{-h}} \frac{1}{x} dx = \frac{e^{-h} - 1}{\bar{x}}$,
for some $\bar{x}$ in $(-h, 0)$
$\frac{e^{-h} - 1}{-h} = \bar{x}$
as $h \to 0^+, -h \to 0^-, \bar{x} \to 0$, then
$\lim_{h \to 0^+} \frac{e^{-h} - 1}{-h} = 0$

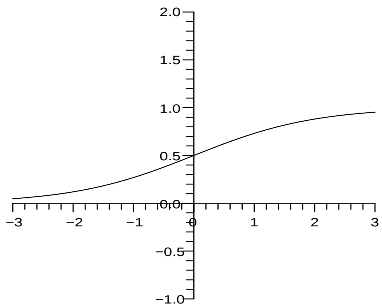
38. $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.
On the other hand
$f'(a) = \lim_{x \to a} \frac{\ln x - \ln a}{x - a}$
$f'(1) = \lim_{x \to 1} \frac{\ln x - \ln 1}{x - 1} = 1$
$\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$
Thus the reciprocal of $\frac{\ln x}{x-1}$ has the same limit,
$\lim_{x \to 1} \frac{x - 1}{\ln x} = 1$
Substituting $x = e^h$, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

39. (a) Given that, $y = \ln(x+1)$ by using a linear approximation.
$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$
For small value of x,
$f(x) \approx f(0) + f'(0)(x - 0)$
$\ln(1 + x) \approx 0 + 1 \cdot (x - 0)$
$\ln(1 + x) \approx x$.

(b) By using area under the curve.
Area the rectangle
$= f(1) \cdot x = x$
Also, $\int_1^{1+x} \frac{1}{t} dt = \ln t|_1^{1+x}$
$= \ln(1 + x) - \ln(1)$
$= \ln(1 + x)$.
As x approaches to zero, we get:
$\ln(1 + x) \approx x$

40. $f(x) = \ln x - 1$
$f'(x) = \frac{1}{x}$
$x_0 = 3$
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{\ln 3 - 1}{\frac{1}{3}}$
$= 6 - 3\ln 3 \approx 2.704163133$
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.718245098$
$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.718281827$
$e \approx 2.718282183$
Three steps are needed to start at $x_0 = 3$ and obtain five digits of accuracy.

41. $f(x) = \dfrac{1}{1+e^{-x}}$



Using $\lim\limits_{x\to\infty} e^{-x} = 0$ we get
$\lim\limits_{x\to\infty} \dfrac{1}{1+e^{-x}} = 1.$
Using $\lim\limits_{x\to-\infty} e^{-x} = \infty$ we get
$\lim\limits_{x\to\infty} \dfrac{1}{1+e^{-x}} = 0.$
The function $f(x)$ is increasing over $(-\infty,\infty)$ and when $x=0$,
$f(0) = \dfrac{1}{1+1} = \dfrac{1}{2}.$

So $g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

The threshold value for $g(x)$ to switch is $x=0$. One way of modifying the function to move the threshold to $x = 4$ is to let $f(x) = \dfrac{1}{1+e^{-(x-4)}}$.

42. $1-(9/10)^{1}0 \approx 0.65132$
$1-(19/20)^{2}0 \approx 0.64151$
$1-(9/10)^{1}0 > 1-(19/20)^{2}0$

The probability of winning is lower.

When taking the limit as $n\to\infty$,

$$\lim_{n\to\infty}\left[1-\left(\frac{n-1}{n}\right)^n\right]$$
$$= 1-\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n$$
$$= 1-\lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n$$
$$= 1-e^{-1}$$

43. $s(x) = x^2\ln(1/x)$
$s'(x) = 2x\ln 1/x + x^2\cdot x\cdot(-1/x^2)$
$= 2x\ln(1/x) - x = x(2\ln(1/x)-1)$

$s'(x) = 0$ gives
$x = 0$ (which is impossible) or
$\ln(1/x) = 1/2, x = e^{-1/2}$.

Since $s'(x)\begin{cases} <0 & \text{if } x < e^{-1/2} \\ >0 & \text{if } x > e^{-1/2} \end{cases}$

The value $x = e^{-1/2}$ maximizes the transmission speed.

44. $\ln\left[\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n\right]$

$$= \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n$$
$$= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)$$
$$= \lim_{n\to\infty}\frac{\ln(1+1/n)}{1/n}$$
$$= \lim_{n\to\infty}\frac{-1/n^2}{-1/n^2(1+1/n)}$$
$$= \lim_{n\to\infty}\frac{1}{1+1/n}$$
$$= 1$$

Ch. 4 Review Exercises

1. $\displaystyle\int(4x^2-3)\,dx = \frac{4}{3}x^3 - 3x + c$

2. $\displaystyle\int(x-3x^5)\,dx = \frac{x^2}{2} - \frac{1}{2}x^6 + c$

3. $\displaystyle\int\frac{4}{x}\,dx = 4\ln|x| + c$

4. $\displaystyle\int\frac{4}{x^2}\,dx = -\frac{4}{x} + c$

5. $\displaystyle\int 2\sin 4x\,dx = -\frac{1}{2}\cos 4x + c$

6. $\displaystyle\int 3\sec^2 x\,dx = 3\tan x + c$

7. $\displaystyle\int(x-e^{4x})\,dx = \frac{x^2}{2} - \frac{1}{4}e^{4x} + c$

8. $\displaystyle\int 3\sqrt{x}\,dx = 2x^{3/2} + c$

9. $\displaystyle\int\frac{x^2+4}{x}\,dx = \int(x+4x^{-1})\,dx$
$\displaystyle= \frac{x^2}{2} + 4\ln|x| + c$

10. $\displaystyle\int\frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + c$

11. $\displaystyle\int e^x(1-e^{-x})\,dx = \int(e^x-1)\,dx$
$= e^x - x + c$

12. $\int e^x(1+e^x)^2\,dx$
$= \int (e^x + 2e^{2x} + e^{3x})\,dx$
$= e^x + e^{2x} + \frac{1}{3}\,e^{3x} + c$

13. Let $u = x^2+4$, then $du = 2x\,dx$ and
$\int x\sqrt{x^2+4}\,dx$
$= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}\,u^{3/2} + c$
$= \frac{1}{3}\,(x^2+4)^{3/2} + c$

14. $\int x(x^2+4)\,dx = \int (x^3+4x)\,dx$
$= \frac{x^4}{4} + 2x^2 + c$

15. Let $u = x^3$, $du = 3x^2\,dx$
$\int 6x^2\cos x^3\,dx = 2\int \cos u\,du$
$= 2\sin u + c = 2\sin x^3 + c$

16. Let $u = x^2$, $du = 2x\,dx$
$\int 4x\sec x^2\tan x^2\,dx$
$= 2\int \sec u\tan u\,du$
$= 2\sec u + c = 2\sec x^2 + c$

17. Let $u = 1/x$, $du = -1/x^2\,dx$
$\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du$
$= -e^u + c = -e^{1/x} + c$

18. Let $u = \ln x$, $du = dx/x$
$\int \frac{\ln x}{x}\,dx = \int u\,du$
$= \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$

19. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$
$= -\ln|\cos x| + c$

20. Let $u = 3x+1, du = 3\,dx$
$\int \sqrt{3x+1}dx = \frac{1}{3}\int u^{1/2}\,du$
$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + c = \frac{2}{9}(3x+1)^{3/2} + c$

21. $f(x) = \int (3x^2+1)\,dx = x^3 + x + c$
$f(0) = c = 2$
$f(x) = x^3 + x + 2$

22. $f(x) = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + c$
$f(0) = -\frac{1}{2} + c = 3$
$c = \frac{7}{2}$
$f(x) = -\frac{1}{2}e^{-2x} + \frac{7}{2}$

23. $s(t) = \int (-32t+10)\,dt$
$= -16t^2 + 10t + c$
$s(0) = c = 2$
$s(t) = -16t^2 + 10t + 2$

24. $v(t) = \int 6\,dt = 6t + c_1$
$v(0) = c_1 = 10$
$v(t) = 6t + 10$
$s(t) = \int (6t+10)\,dt = 3t^2 + 10t + c_2$
$s(0) = c_2 = 0$
$s(t) = 3t^2 + 10t$

25. $\sum_{i=1}^{6}(i^2+3i)$
$= (1^2+3\cdot1) + (2^2+3\cdot2) + (3^2+3\cdot3)$
$+ (4^2+3\cdot4) + (5^2+3\cdot5) + (6^2+3\cdot6)$
$= 4 + 10 + 18 + 28 + 40 + 54$
$= 154$

26. $\sum_{i=1}^{12} i^2 = 650$

27. $\sum_{i=1}^{100}(i^2-1)$
$= \sum_{i=1}^{100} i^2 - \sum_{i=1}^{100} 1$
$= \frac{100(101)(201)}{6} - 100$
$= 338,250$

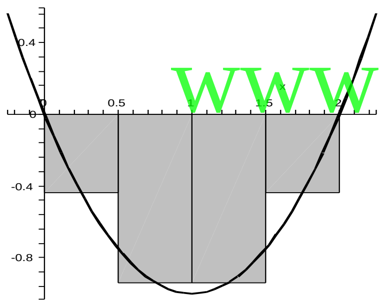
28. $\sum_{i=1}^{100}(i^2+2i)$
$= \sum_{i=1}^{100} i^2 + 2\cdot\sum_{i=1}^{100} i$
$= \frac{100(101)(201)}{6} + 100(101)$
$= 348,450$

29. $\frac{1}{n^3}\sum_{i=1}^{n}(i^2-i)$
$= \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - \cdot\sum_{i=1}^{n} i\right)$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right)$$
$$= \frac{(n+1)(2n+1)}{6n^2} - \frac{n+1}{2n^2}$$
$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n}(i^2 - i)$$
$$= \lim_{n\to\infty}\left(\frac{(n+1)(2n+1)}{6n^2} - \frac{n+1}{2n^2}\right)$$
$$= \frac{2}{6} - 0 = \frac{1}{3}$$

30. Evaluation points: $0.25, 0.75, 1.25, 1.75$

Riemann sum $= \Delta x \sum_{i=1}^{n} f(c_i)$
$$= \frac{2}{4}\sum_{i=1}^{4}(c_i^2 - 2c_i)$$
$$= \frac{1}{2}\big[(0.25^2 - 2\cdot 0.25) + (0.75^2 - 2\cdot 0.75)$$
$$+(1.25^2 - 2\cdot 1.25) + (1.75^2 - 2\cdot 1.75)\big]$$
$$= -2.75$$



31. Riemann sum $= \frac{2}{8}\sum_{i=1}^{8} c_i^2 = 2.65625$

32. Riemann sum $= \frac{2}{8}\sum_{i=1}^{8} c_i^2 = 0.6875$

33. Riemann sum $= \frac{3}{8}\sum_{i=1}^{8} c_i^2 \approx 4.668$

34. Riemann sum $= \frac{1}{8}\sum_{i=1}^{8} c_i^2 \approx 0.6724$

35.

(a) Left-endpoints:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{8}(f(0) + f(.2) + f(.4)$$
$$+ f(.6) + f(.8) + f(1) + f(1.2)$$
$$+ f(1.4))$$
$$= \frac{1}{5}(1 + 1.4 + 1.6 + 2 + 2.2 + 2.4$$
$$+ 2 + 1.6)$$
$$= 2.84$$

(b) Right-endpoints:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{8}(f(.2) + f(.4) + f(.6)$$
$$+ f(.8) + f(1) + f(1.2) + f(1.4)$$
$$+ f(1.6))$$
$$= \frac{1}{5}(1.4 + 1.6 + 2 + 2.2 + 2.4$$
$$+ 2 + 1.6 + 1.4)$$
$$= 2.92$$

(c) Trapezoidal Rule:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{2(8)}[f(0) + 2f(.2) + 2f(.4)$$
$$+ 2f(.6) + 2f(.8) + 2f(1)$$
$$+ 2f(1.2) + 2f(1.4) + f(1.6)]$$
$$= 2.88$$

(d) Simpson's Rule:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{3(8)}[f(0) + 4f(.2) + 2f(.4)$$
$$+ 4f(.6) + 2f(.8) + 4f(1)$$
$$+ 2f(1.2) + 4f(1.4) + f(1.6)]$$
$$\approx 2.907$$

36.

(a) Left-endpoints:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx (0.4)[f(1.0) + f(1.4) + f(1.8)$$
$$+ f(2.2) + f(2.6) + f(3.0)$$
$$+ f(3.4) + f(3.8)]$$
$$= (0.4)(4.0 + 3.4 + 3.6 + 3.0$$
$$+ 2.6 + 2.4 + 3.0 + 3.6)$$
$$= 10.24$$

(b) Right-endpoints:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx (0.4)[f(1.4) + f(1.8) + f(2.2)$$
$$+ f(2.6) + f(3.0) + f(3.4)$$
$$+ f(3.8) + f(4.2)]$$
$$= (0.4)(3.4 + 3.6 + 3.0 + 2.6$$

$+ 2.4 + 3.0 + 3.6 + 3.4)$
$= 10.00$

(c) Trapezoidal Rule:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx \frac{0.4}{2}[f(1.0) + 2f(1.4) + 2f(1.8)$$
$$+ 2f(2.2) + 2f(2.6) + 2f(3.0)$$
$$+ 2f(3.4) + 2f(3.8) + f(4.2)]$$
$$= (0.2)[4.0 + 2(3.4) + 2(3.6)$$
$$+ 2(3.0) + 2(2.6) + 2(2.4)$$
$$+ 2(3.0) + 2(3.6) + 3.4]$$
$$= 10.12$$

(d) Simpson's Rule:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx \frac{0.4}{3}[f(1.0) + 4f(1.4) + 2f(1.8)$$
$$+ 4f(2.2) + 2f(2.6) + 4f(3.0)$$
$$+ 2f(3.4) \quad + 4f(3.8) + f(4.2)]$$
$$= \frac{0.4}{3}[4.0 + 4(3.4) + 2(3.6)$$
$$+ 4(3.0) + 2(2.6) + 4(2.4)$$
$$+ 2(3.0) + 4(3.6) + 3.4]$$
$$\approx 10.05333$$

37. See Example 7.10.

Simpson's Rule is expected to be most accurate.

38. In this situation, the Midpoint Rule will be less than the actual integral. The Trapezoid Rule will be an overestimate.

39. We will compute the area A_n of n rectangles using right endpoints. In this case $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$
$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} 2\cdot\left(\frac{i}{n}\right)^2$$
$$= \frac{2}{n^3}\sum_{i=1}^{n} i^2$$
$$= \left(\frac{2}{n^3}\right)\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{(n+1)(2n+1)}{3n^2}$$
Now, to find the integral, we take the limit:
$$\int_0^1 x^2\,dx = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{3n^2}$$
$$= \frac{2}{3}$$

40. We will compute the area A_n of n rectangles using right endpoints. In this case $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$
$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{2}{n}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2 + 1$$
$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$
$$= \left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6} + \left(\frac{2}{n}\right)n$$
$$= \frac{4(n+1)(2n+1)}{3n^2} + 2$$
Now, to find the integral, we take the limit:
$$\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{4(n+1)(2n+1)}{3n^2} + 2\right)$$
$$= \frac{8}{3} + 2 = \frac{14}{3}$$

41. Area $= \int_0^3 (3x - x^2)\,dx$
$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \frac{9}{2}$$

42. Area
$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$
$$- \int_1^2 (x^3 - 3x^2 + 2x)\,dx$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

43. The velocity is always positive, so distance traveled is equal to change in position.
$$\text{Dist} = \int_1^2 (40 - 10t)\,dt$$
$$= (40t - 5t^2)\Big|_1^2 = 25$$

44. The velocity is always positive, so distance traveled is equal to change in position.
$$\text{Dist} = \int_0^2 20e^{-t/2}\,dt = (-40e^{-t/2})\Big|_0^2$$
$$= 40(-e^{-1} + 40) \approx 25.2848$$

45. $f_{ave} = \dfrac{1}{2}\displaystyle\int_0^2 e^x\,dx = \dfrac{e^2-1}{2} \approx 3.19$

46. $f_{ave} = \dfrac{1}{4}\displaystyle\int_0^4 (4x - x^2)\,dx = \dfrac{8}{3}$

47. $\displaystyle\int_0^2 (x^2-2)\,dx = \left(\frac{x^3}{3} - 2x\right)\Big|_0^2 = -\frac{4}{3}$

48. $\displaystyle\int_{-1}^1 (x^3-2x)\,dx = \left(\frac{x^4}{4} - x^2\right)\Big|_{-1}^1 = 0$

49. $\displaystyle\int_0^{\pi/2} \sin 2x\,dx = -\frac{1}{2}\cos 2x\Big|_0^{\pi/2} = 1$

50. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = 1$

51. $\displaystyle\int_0^{10} (1-e^{-t/4})\,dt$
$= \left(t + 4e^{-t/4}\right)\Big|_0^{10} = 6 + 4e^{-5/2}$

52. $\displaystyle\int_0^1 te^{-t^2}\,dt$
$= \left(-\dfrac{1}{2}e^{-t^2}\right)\Big|_0^1 = -\dfrac{1}{2}(e^{-1}-1)$

53. $\displaystyle\int_0^2 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|\Big|_0^2$
$= \dfrac{\ln 5}{2}$

54. $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx = \left(\frac{\ln^2 x}{2}\right)\Big|_1^2 = \frac{\ln^2 2}{2}$

55. $\displaystyle\int_0^2 x\sqrt{x^2+4}\,dx$
$= \left(\dfrac{1}{2}\cdot\dfrac{2}{3}\cdot(x^2+4)^{3/2}\right)\Big|_0^2$
$= \dfrac{16\sqrt{2}-8}{3}$

56. $\displaystyle\int_0^2 x(x^2+1)\,dx$
$= \left(\dfrac{1}{4}(x^2+1)^2\right)\Big|_0^2 = 6$

57. $\displaystyle\int_0^1 (e^x-2)^2\,dx = \int_0^1 (e^{2x-4e^x+4})\,dx$
$= \left(\dfrac{1}{2}e^{2x} - 4e^x + 4x\right)\Big|_0^2$
$= \left(\dfrac{e^2}{2} - 4e + 4\right) - \left(\dfrac{1}{2} - 4\right)$
$= \dfrac{e^2}{2} - 4e + \dfrac{15}{2}$

58. $\displaystyle\int_{-\pi}^{\pi} \cos(x/2)\,dx$
$= (2\sin(x/2))\Big|_{-\pi}^{\pi} = 4$

59. $f'(x) = \sin x^2 - 2$

60. $f'(x) = \sqrt{(x^2)^2+1}\cdot 2x$

61.

a) Midpoint Rule:
$\displaystyle\int_0^1 \sqrt{x^2+4}\,dx$
$\approx \dfrac{1-0}{4}\left[f\left(\dfrac{1}{8}\right) + f\left(\dfrac{3}{8}\right)\right.$
$\left. + f\left(\dfrac{5}{8}\right) + f\left(\dfrac{7}{8}\right)\right]$
≈ 2.079

b) Trapezoidal Rule:
$\displaystyle\int_0^1 \sqrt{x^2+4}\,dx$
$\approx \dfrac{1-0}{2(4)}\left[f(0) + 2f\left(\dfrac{1}{4}\right)\right.$
$+ 2f\left(\dfrac{1}{2}\right) + 2f\left(\dfrac{3}{4}\right)$
$\left. + f(1)\right]$
≈ 2.083

c) Simpson's Rule:
$\displaystyle\int_0^1 \sqrt{x^2+4}\,dx$
$\approx \dfrac{1-0}{3(4)}\left[f(0) + 4f\left(\dfrac{1}{4}\right)\right.$
$\left. + 2f\left(\dfrac{1}{2}\right) + 4f\left(\dfrac{3}{4}\right) + f(1)\right]$
≈ 2.080

62.

a) Midpoint Rule:
$\displaystyle\int_0^2 e^{-x^2/4}\,dx$
$\approx \dfrac{2}{4}[f(0.25) + f(0.75)$
$+ f(1.25) + f(1.75)]$
≈ 1.497494

b) Trapezoidal Rule:
$\displaystyle\int_0^2 e^{-x^2/4}\,dx$
$\approx \dfrac{2}{8}[f(0) + 2f(.5) + 2f(1)$

$+ 2f(1.5) + f(2)]$
≈ 1.485968

c) Simpson's Rule:

$$\int_0^2 e^{-x^2/4}\,dx$$
$$\approx \frac{2}{12}[f(0) + 4f(.5) + 2f(1)$$
$$+ 4f(1.5) + f(2)]$$
$$\approx 1.493711$$

63.

n	Midpoint	Trapezoid	Simpson's
20	2.08041	2.08055	2.08046
40	2.08045	2.08048	2.08046

64.

n	Midpoint	Trapezoid	Simpson's
20	1.493802	1.493342	1.493648
40	1.493687	1.493572	1.493648

65. Consider $u = \tanh\left(\frac{t}{2}\right) = \dfrac{\sinh\left(\frac{t}{2}\right)}{\cosh\left(\frac{t}{2}\right)}$

$$= \frac{\left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2}\right)}{\left(\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2}\right)} = \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)}$$

therefore $\dfrac{1-u^2}{1+u^2} = \dfrac{1 - \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^2}{1 + \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^2}$

$$= \frac{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2 - \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^2}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2 + \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^2}$$
$$= \frac{2\left(e^t + e^{-t}\right)}{4} = \cosh t\,,$$

similarly, $\dfrac{2u}{1+u^2} = \dfrac{2\frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)}}{1 + \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^2}$

$$= \frac{2\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2 + \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^2}$$
$$= \frac{2\left(e^t - e^{-t}\right)}{4} = \sinh t$$

(a) $\displaystyle\int \frac{1}{\sinh t + \cosh t}dt$

$$= \int \frac{1}{\frac{2u}{(1-u^2)} + \frac{(1+u^2)}{(1-u^2)}}du$$
(Put: $u = \tanh(t/2)$)
$$= \int \frac{(1-u^2)}{(1+u)^2}du$$
$$= \int \left(\frac{1-u}{1+u}\right)du$$
$$= \int \left(\frac{2}{1+u} - 1\right)du$$
$$= 2\ln(1+u) - u$$
$$= 2\ln(1+\tanh(t/2)) - \tanh(t/2)$$

(b) $\displaystyle\int \frac{\sinh t + \cosh t}{1 + \cosh t}dt$

$$= \int \frac{\frac{2u}{(1-u^2)} + \frac{(1+u^2)}{(1-u^2)}}{1 + \frac{(1+u^2)}{(1-u^2)}}du$$
$$= \int \frac{(1+u)^2}{2}du$$
$$= \frac{1}{2}\left(\frac{(1+u)^3}{3}\right)$$
$$= \frac{(1+\tanh(t/2))^3}{6}$$